Large-*n* expansion for *m*-axial Lifshitz points

2005 J. Phys.: Condens. Matter 17 S1947

(http://iopscience.iop.org/0953-8984/17/20/020)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS: CONDENSED MATTER

J. Phys.: Condens. Matter **17** (2005) S1947–S1972 doi:10.1088/0953-8984/17/20/020

Large-n expansion for m-axial Lifshitz points

M A Shpot[1,2], **Yu M Pis'mak**[2,3] **and H W Diehl**[2]

[1] Institute for Condensed Matter Physics, 79011 Lviv, Ukraine
[2] Fachbereich Physik, Universität Duisburg-Essen, Campus Essen, D-45117 Essen, Federal Republic of Germany
[3] State University of Sankt-Petersburg, 198504 Sankt-Petersburg, Russia

E-mail: shpot@ph.icmp.lviv.ua

Received 15 December 2004, in final form 17 January 2005
Published 6 May 2005
Online at stacks.iop.org/JPhysCM/17/S1947

Abstract

The large-n expansion is developed for the study of critical behaviour of d-dimensional systems at m-axial Lifshitz points with an arbitrary number m of modulation axes. The leading nontrivial contributions to $O(1/n)$ are derived for the two independent correlation exponents η_{L2} and η_{L4}, and the related anisotropy index θ. The series coefficients of these $1/n$ corrections are given for general values of m and d with $0 \leqslant m \leqslant d$ and $2 + m/2 < d < 4 + m/2$ in the form of integrals. For special values of m and d such as $(m, d) = (1, 4)$, they can be computed analytically, but in general their evaluation requires numerical means. The $1/n$ corrections are shown to reduce in the appropriate limits to those of known large-n expansions for the case of d-dimensional isotropic Lifshitz points and critical points, respectively, and to be in conformity with available dimensionality expansions about the upper and lower critical dimensions. Numerical results for the $1/n$ coefficients of η_{L2}, η_{L4} and θ are presented for the physically interesting case of a uniaxial Lifshitz point in three dimensions, as well as for some other choices of m and d. A universal coefficient associated with the energy-density pair correlation function is calculated to leading order in $1/n$ for general values of m and d.

1. Introduction

Systems exhibiting critical behaviour can be divided into universality classes such that all members of a given class have the same universal critical properties (Fisher 1983). The universality classes are represented by field theories, such as the n-component ϕ^4 models in d space dimensions, which are nontrivial whenever d is less than the upper critical dimension d^* above which Landau theory holds. Proper analyses of such field theories usually require sophisticated tools such as renormalization group approaches, elaborate numerical simulations or a combination of the two (Wilson and Kogut 1974, Domb and Green 1976).

The commonly employed successful analytical methods are dimensionality expansions about the upper and lower critical dimensions d^* and d_ℓ (Wilson and Fisher 1972, Polyakov 1975, Brézin and Zinn-Justin 1976a, 1976b, Bardeen *et al* 1976), the massive renormalization group (RG) approach in fixed dimensions (Parisi 1980) and the $1/n$ expansion in inverse powers of the number n of order parameter components (Ma 1973, Abe 1973). Remarkably precise estimates of critical indices and other universal quantities of three-dimensional systems have been obtained both via $\epsilon = d^* - d$ expansions and the massive RG approach at fixed d by computing perturbation series to sufficiently high orders and resumming them (Guida and Zinn-Justin 1998, Pelissetto and Vicari 2002, Privman *et al* 1991). Expansions about the lower critical dimension d_ℓ, on the other hand, seem to have a more modest potential for precise estimates, unless they are combined with information from other sources.

In this paper we will be concerned with the $1/n$ expansion. Our aim is to develop this approach for the study of critical behaviour at m-axial Lifshitz points. A Lifshitz point (LP) is a multicritical point at which a disordered phase meets both a spatially uniform ordered and a spatially modulated ordered phase (Hornreich *et al* 1975a, Hornreich 1980, Selke 1992, Diehl 2002, 2004). The disordered phase is separated from the two ordered ones by a line of critical temperatures $T_c = T_c(g)$, depending on a thermodynamic nonordering field g, such as pressure or a ratio of an antiferromagnetic and a ferromagnetic coupling. The LP, located at $T_L = T_c(g_L)$, divides this critical line into two sections. In the modulated ordered phase, the wavevector associated with the modulation, $\boldsymbol{q}_{\text{mod}} = \boldsymbol{q}_{\text{mod}}(T, g)$, varies with g and temperature T. Without loss of generality, we can consider the ferromagnetic case. Then uniform order corresponds to $\boldsymbol{q}_{\text{mod}} = \boldsymbol{0}$, and $\boldsymbol{q}_{\text{mod}}(T, g)$ vanishes at the LP. The LP is called m-axial if the wavevector instability that sets in at the LP occurs in an m-dimensional subspace of d-dimensional space, i.e. $\boldsymbol{q}_{\text{mod}} \in \mathbb{R}^m$ with $0 \leqslant m \leqslant d$. The limiting values $m = 0$ and d correspond to the cases of a usual critical point (CP) and the isotropic LP, respectively.

As is well known, the large-n expansion leads to self-consistent equations (Abe 1973, Brézin *et al* 1976, Vasiliev *et al* 1981a). These are considerably more difficult to handle than ordinary perturbation expansions, which are the essential input required in the alternative approaches mentioned above (dimensionality expansions, massive RG approach). For this reason the available series expansions in $1/n$ are restricted to low orders, even for the much simpler case of a conventional CP. Furthermore, $1/n$ expansions normally converge slowly[4]. On the other hand, this technique has a number of very attractive features. One is its capability of treating fluctuation effects in a systematic, nonperturbative manner[5]. Another is that it can be applied for any fixed value of d between d_ℓ and d^*. No additional expansion in a small parameter such as $\epsilon = d^* - d$ is required. Owing to these appealing properties the $1/n$ expansion continues to be an important tool for the analysis of nontrivial field theories. The spectrum of problems to which it has recently been applied and made significant contributions is impressively rich. It ranges from the study of classical spin models (Brézin 1993, Vasiliev 1998, Zinn-Justin 1996, Moshe and Zinn-Justin 2003, Campostrini *et al* 1998, Pelissetto *et al* 2001, Gracey 2002a, 2002b), critical behaviour in bounded systems (McAvity and Osborn 1995), the physics of disordered elastic media (Le Doussal and Wiese 2003, 2004) and models of stochastic surface growth (Doherty *et al* 1994) to problems of high-energy physics (Aharony *et al* 2000, Moshe and Zinn-Justin 2003) and quantum phase transitions (Franz *et al* 2002).

[4] According to some recent results (Baym *et al* 2000, Arnold and Tomášik 2000), the shift of the critical temperature of a dilute Bose gas seems to be an exception to this rule.

[5] A familiar numerical RG scheme having this capability employs the so-called 'exact' RG equations (Wilson and Kogut 1974). Its only application to critical behaviour at LPs we are aware of is the recent work by Bervillier (2004). This deals with the case of a uniaxial LP with $n = 1$ in $d = 3$ dimensions and uses the lowest ('local potential') approximation.

Unfortunately, for critical behaviour at LPs, hardly any results have yet been worked out by means of it. Previous large-n analyses for general values of $m \in [0, d]$ (Kalok and Obermair 1976, Hornreich *et al* 1977, Mukamel and Luban 1978, Frachebourg and Henkel 1993) were restricted to the limit $n \to \infty$. To our knowledge, the only exceptions in which corrections to order $1/n$ were computed deal exclusively with the case $m = d$ of a d-axial LP (Hornreich *et al* 1975b, Nicoll *et al* 1976, Inayat-Hussain and Buckingham 1990).

This lack of results for general m is due to the complications arising from the combination of two special features such systems have: (i) the *anisotropic* nature of scale invariance they exhibit at LPs; and (ii) the *unusually complicated form* of the two-point scaling functions they have in position space already at the level of Landau theory (Diehl and Shpot 2000, Shpot and Diehl 2001). Anisotropic scale invariance means that coordinate separations $\mathrm{d}z_\alpha$ along some directions scale as nontrivial powers of their counterparts $\mathrm{d}r_\beta$ along the remaining orthogonal ones, i.e. $\mathrm{d}z_\alpha \sim (\mathrm{d}r_\beta)^\theta$, where θ, called the anisotropy exponent, differs from 1. Of course, anisotropic scale invariance is a feature encountered also in studies of dynamical critical behaviour at usual CPs, where time t scales as a nontrivial power of the spatial separations, and for which analyses to $\mathrm{O}(1/n)$ can be found in the literature (see e.g. Halperin *et al* (1972); for more recent uses of the $1/n$ expansion in dynamics, see e.g. Bray (2002) or Moshe and Zinn-Justin (2003)). It is the combination with (ii) that makes consistent treatments of fluctuation effects so hard for critical behaviour at m-axial LPs. The very same difficulties had prevented the performance of full two-loop RG analyses within the framework of the $\epsilon = d^* - d$ expansion for decades, and produced controversial $\mathrm{O}(\epsilon^2)$ results (Mukamel 1977, Sak and Grest 1978, Hornreich and Bruce 1978, Hornreich 1980, Mergulhão and Carneiro 1999) until such a RG analysis for general m was finally accomplished in Diehl and Shpot (2000) and Shpot and Diehl (2001)[6].

In order to overcome the above-mentioned technical difficulties, we shall adapt and extend the elegant technique suggested by Vasiliev *et al* (1981a) and utilized in subsequent work (Vasiliev *et al* 1981b) to compute the $1/n$ expansion of the standard critical exponents η and ν for the case $m = 0$ of a CP to $\mathrm{O}(1/n^2)$. (In Vasiliev *et al* (1981c), the authors managed to compute η even to $\mathrm{O}(1/n^3)$ by means of a conformal bootstrap method.) We proceed as follows.

We start in section 2 by specifying the expected scaling forms of the required two-point functions. These are employed in section 3 to solve the resulting self-consistent equations to the appropriate order $1/n$. Matching the anticipated asymptotic large-distance behaviour of the cumulants with the solutions of these equations yields consistency conditions. From these, the $\mathrm{O}(1/n)$ terms of the two independent correlation exponents η_{L2} and η_{L4} can be determined for general values of m in the form of finite, numerically computable integrals.

In section 4 we verify that these coefficients reduce in the limits $m \to 0$ and $m \to d$ to the analytical expressions for the CP case and the isotropic LP obtained in Ma (1973), Abe and Hikami (1973) and Hornreich *et al* (1975b), respectively. In section 5 we consider the behaviour of our $\mathrm{O}(1/n)$ results as $\epsilon = 4 + m/2 - d \to 0$, and explicitly reproduce the large-n limits of the $\mathrm{O}(\epsilon^2)$ coefficients of Diehl and Shpot (2000) and Shpot and Diehl (2001) for the correlation exponents η_{L2} and η_{L4} in the case $m = 2$ of a biaxial LP. The consistency of our results with known expansions about the lower critical dimension $d_\ell = 2 + m/2$ is demonstrated in section 6.

In section 7 we first consider the special cases $(m, d) = (1, 4)$ and $(4, 5)$. For the former, we derive fully analytical expressions for the $1/n$ coefficients of the correlation exponents and

[6] For a discussion and critical assessment of recent work (de Albuquerque and Leite (2001), Leite (2003)) giving alternative results, see Diehl and Shpot (2001, 2003).

the anisotropy exponent. The latter case requires some numerical work, even though part of the calculations can also be done analytically. Next, we turn to the physically interesting case $(m, d) = (1, 3)$ of a uniaxial LP in three dimensions and present numerical results for the coefficients of the $1/n$ contributions. In section 8 we compute the universal scaling function of the energy-density pair correlation function and a related universal amplitude to leading order in $1/n$. We give analytical results for the expansion coefficient of the latter for general values of (m, d) with $d_\ell < d < d^*$. Our results are briefly discussed and put in perspective in the closing section 9. Finally, there are two appendices describing technical details.

2. Scaling properties of the two-point functions

We wish to consider the theory of an n-component vector field $\boldsymbol{\phi}(\boldsymbol{x}) = (\phi_1(\boldsymbol{x}), \ldots, \phi_n(\boldsymbol{x}))$ in d space dimensions. Writing $\boldsymbol{x} = (\boldsymbol{r}, \boldsymbol{z})$, we decompose the position vector $\boldsymbol{x} \in \mathbb{R}^d$ into a 'parallel' component $\boldsymbol{z} \in \mathbb{R}^m$ and a 'perpendicular' one $\boldsymbol{r} \in \mathbb{R}^{\bar{m}}$, where $\bar{m} = d - m$ is the codimension of m. Likewise, $\partial_r \equiv \partial/\partial \boldsymbol{r}$ and $\partial_z = \partial/\partial \boldsymbol{z}$ denote the corresponding components of the gradient operator $\nabla = (\partial_r, \partial_z)$. The Hamiltonian of the model we will investigate is given by

$$\mathcal{H}[\phi] = \mathcal{H}_0[\phi] + \mathcal{H}_{\text{int}}[\phi], \tag{1}$$

with the O(n)-symmetric free and interacting parts

$$\mathcal{H}_0[\phi] = \frac{1}{2}\int \mathrm{d}^d x \, \left[(\partial_r \phi)^2 + (\partial_z^2 \phi)^2 + \mathring{\rho}\, (\partial_z \phi)^2 + \mathring{\tau}\, \phi^2\right] \tag{2}$$

and

$$\mathcal{H}_{\text{int}}[\phi] = \frac{\lambda}{8}\int \mathrm{d}^d x \, \phi^4, \tag{3}$$

respectively.

In order for the model to have a LP, the dimension d must exceed $d_\ell(m, n)$, the lower critical dimension, which is believed to be $d_\ell^{\mathrm{O}(n)}(m) = 2 + m/2$ for the case $n > 1$ of continuous O(n) symmetry we are concerned with here (Hornreich *et al* 1975a, Grest and Sak 1978, Diehl 2002).

A cautionary remark should be added here. The arguments of Hornreich *et al* (1975a) and the expansion about the dimension $d = 2 + m/2$ (Grest and Sak 1978) actually show only that the *homogeneous ordered* phase becomes thermodynamically unstable to low-energy excitations at temperatures $T > 0$ whenever $d \leqslant 2 + m/2$. To establish that a LP exists for $d > 2 + m/2$, one would have to prove additionally the existence of a *modulated ordered* phase that is separated via a *second-order* line from the disordered phase. According to an argument given by Garel and Pfeuty (1976) one expects the transition from the disordered to the modulated ordered phase to be described by a $2n$-component ϕ^4 model whose Hamiltonian has O(n) × O(n) symmetry. Although our results are consistent with the existence of a LP for $n > 1$ and $d > 2+m/2$, we cannot rule out the possibility that for some values of n, fluctuations might change this transition into a discontinuous one. Such a scenario apparently occurs in the $n = 1$ case of ternary mixtures of A and B homopolymers and AB diblock copolymers, where fluctuations were found to transform the continuous mean-field transition between the disordered and lamellar phases into a discontinuous one (see e.g. Düchs *et al* (2003) and its references), in accordance with general arguments given by Brazovskiĭ (1975).

Whenever a LP exists, we denote the values of $\mathring{\tau}$ and $\mathring{\rho}$ at which it is located by $\mathring{\tau}_{\mathrm{L}}$ and $\mathring{\rho}_{\mathrm{L}}$, respectively.

To derive the large-n expansion, it is convenient to introduce an auxiliary scalar field $\psi = \psi(\boldsymbol{x})$ and rewrite the interaction part $\mathcal{H}_{\text{int}}[\phi]$ as (Brézin *et al* 1976, Vasiliev *et al* 1981a, Vasiliev 1998, Moshe and Zinn-Justin 2003)

$$e^{-\mathcal{H}_{\text{int}}[\phi]} = \text{const} \int \mathcal{D}\psi \, e^{-\frac{1}{2}\int d^d x \, [\psi^2 - i\sqrt{\lambda}\,\phi^2 \psi]}. \tag{4}$$

The full Hamiltonian defined by equations (1)–(3) then becomes

$$\mathcal{H}[\phi, \psi] = \mathcal{H}_0[\phi] + \tfrac{1}{2} \int d^d x \, [\psi^2 - i\sqrt{\lambda}\,\phi^2\,\psi], \tag{5}$$

up to an irrelevant additive constant.

In view of this reformulation of the model it is natural to consider correlation functions involving besides the fields ϕ_a with $a = 1, \ldots, n$ also ψ. We need in particular the two-point functions. Let us consider the disordered phase. Since in it the O(n) symmetry of the Hamiltonian (5) is not spontaneously broken, the mixed correlation functions $\langle \phi_a \, \psi \rangle$ vanish. Furthermore, invariance under translations and rotations in the position subspaces $\mathbb{R}^m$ and $\mathbb{R}^{\bar{m}}$ implies

$$\left\langle \phi_{a_1}(\boldsymbol{x} + \boldsymbol{x}') \, \phi_{a_2}(\boldsymbol{x}') \right\rangle = \delta_{a_1 a_2} \, G_\phi(r, z) \tag{6}$$

and

$$\langle \psi(\boldsymbol{x} + \boldsymbol{x}') \, \psi(\boldsymbol{x}') \rangle = G_\psi(r, z), \tag{7}$$

where $r \equiv |\boldsymbol{r}|$ and $z \equiv |\boldsymbol{z}|$ are the lengths of the perpendicular and parallel components of $\boldsymbol{x} = (\boldsymbol{r}, \boldsymbol{z})$. Writing the wavevector conjugate to $\boldsymbol{x}$ as $\boldsymbol{k} = (\boldsymbol{p}, \boldsymbol{q})$, we introduce Fourier transforms $\tilde{G}_{\phi,\psi}(p, q)$ of the two-point functions $G_{\phi,\psi}(r, z)$ via

$$G_{\phi,\psi}(r, z) = \int_{\boldsymbol{k}}^{(d)} \tilde{G}_{\phi,\psi}(p, q) \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{8}$$

where

$$\int_{\boldsymbol{k}}^{(d)} \equiv \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^d} = \int_{\boldsymbol{p}}^{(\bar{m})} \int_{\boldsymbol{q}}^{(m)} \tag{9}$$

is convenient shorthand.

At a LP, the correlation functions $G_{\phi,\psi}$ are expected to decay as powers of r and z in the long-distance limits $r, z \to \infty$ provided that $d_\ell^{\mathrm{O}(n)} < d < d^*$. Furthermore, they should be anisotropically scale invariant. Thus, for appropriate choices of the values of the scaling dimensions Δ_ϕ and Δ_ψ of the fields ϕ_a and ψ and of the anisotropy exponent θ, the limits

$$\lim_{b\to\infty} b^{2\Delta_\phi} \, G_\phi(br, b^\theta z) = G_\phi^{(\text{as})}(r, z) \tag{10}$$

and

$$\lim_{b\to\infty} b^{2\Delta_\psi} \, G_\psi(br, b^\theta z) = G_\psi^{(\text{as})}(r, z) \tag{11}$$

should exist, yielding nontrivial asymptotic functions $G_{\phi,\psi}^{(\text{as})}(r, z)$. Similar results must hold for their Fourier transforms, namely

$$\lim_{b\to\infty} b^{-2\tilde{\Delta}_{\phi,\psi}} \, \tilde{G}_{\phi,\psi}^{(\text{as})}(b^{-1}p, b^{-\theta}q) = \tilde{G}_{\phi,\psi}^{(\text{as})}(p, q), \tag{12}$$

where the scaling dimensions are given by

$$2\tilde{\Delta}_{\phi,\psi} = d - m/2 - 2\Delta_{\phi,\psi}. \tag{13}$$

The asymptotic functions (12) are generalized homogeneous functions satisfying

$$\tilde{G}^{(\mathrm{as})}_{\phi,\psi}(p,q) = p^{-2\tilde{\Delta}_{\phi,\psi}}\,\tilde{G}^{(\mathrm{as})}_{\phi,\psi}(1,p^{-\theta}q) = q^{-2\tilde{\Delta}_{\phi,\psi}/\theta}\,\tilde{G}^{(\mathrm{as})}_{\phi,\psi}\big(q^{-1/\theta}p,1\big). \tag{14}$$

The exponents governing the momentum dependences of $\tilde{G}^{(\mathrm{as})}_{\phi}$ are conventionally written as

$$2\tilde{\Delta}_{\phi} = 2-\eta_{\mathrm{L}2}, \qquad 2\tilde{\Delta}_{\phi}/\theta = 4-\eta_{\mathrm{L}4}, \tag{15}$$

which defines the correlation exponents $\eta_{\mathrm{L}2}$ and $\eta_{\mathrm{L}4}$.

In the limit $n\to\infty$, the momentum-space propagators at the LP become

$$\tilde{G}^{(0)}_{\phi}(p,q) = \left(p^2+q^4\right)^{-1}, \qquad \tilde{G}^{(0)}_{\psi}(p,q) = 1, \tag{16}$$

so that the exponents take their known spherical-model values

$$\eta^{(0)}_{\mathrm{L}2} = \eta^{(0)}_{\mathrm{L}4} = 0, \qquad \theta^{(0)} = 1/2. \tag{17}$$

For finite n, fluctuations modify both the propagators and the critical exponents. Making the ansätze

$$\eta_{\mathrm{L}2}(m,d) = \frac{\eta^{(1)}_{\mathrm{L}2}}{n} + \mathrm{O}(n^{-2}), \qquad \theta(m,d) = \frac{1}{2} + \frac{\theta^{(1)}}{n} + \mathrm{O}(n^{-2}), \tag{18}$$

we shall determine the coefficients $\eta^{(1)}_{\mathrm{L}2}$ and $\theta^{(1)}$ of the $1/n$ corrections in what follows. For $\eta_{\mathrm{L}4}$ the $1/n$ expansion is analogous to that of $\eta_{\mathrm{L}2}$; the scaling relation (15) implies that its $1/n$ coefficient is given by

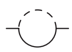
$$\eta^{(1)}_{\mathrm{L}4} = 2\big(\eta^{(1)}_{\mathrm{L}2} + 4\theta^{(1)}\big). \tag{19}$$

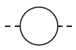
3. Self-consistent equations

The self-consistent equations that must be solved in order to determine the $\mathrm{O}(1/n)$ corrections to the correlation exponents can be derived by well-known steepest descent or diagrammatic methods, which are amply described in the literature (see e.g. Ma (1973), Abe and Hikami (1973), Brézin *et al* (1976), Vasiliev *et al* (1981a), Vasiliev (1998), Moshe and Zinn-Justin (2003)). Solving them is, however, another story: owing to the anisotropic character of scale invariance at LPs, the correlation functions $\tilde{G}_{\phi}$ and $\tilde{G}_{\psi}$ involve scaling *functions*, rather than being simple powers. Finding solutions to these equations is therefore a nontrivial task.

We use an appropriately modified variant of the technique advocated by Vasiliev *et al* (1981a, 1981b) and employed to compute the critical exponents η and ν of the standard n-component $|\phi|^4$ model up to order n^{-2}. The crux of this method is to make a scaling ansatz for the asymptotic large-length-scale form of the solutions and determine the exponents it involves from the conditions implied by the self-consistent equations.

These equations can be written as

$$G^{-1}_{\phi} = G^{(0)-1}_{\phi} - \Sigma_{\phi} = G^{(0)-1}_{\phi} - \tag{20}$$

$- \ldots,$

$$G^{-1}_{\psi} = G^{(0)-1}_{\psi} - \Sigma_{\psi} = G^{(0)-1}_{\psi} - \tag{21}$$

$- \ldots,$

where the full and broken lines denote the full propagators (6) and (7), respectively.

Since the propagator (6) is proportional to $\delta_{a_1a_2}$, each closed loop of G_{ϕ} lines produces a factor n. We assume that $G_{\phi} = \mathrm{O}(1)$ as $n\to\infty$. Then it follows from equation (21) that $G^{-1}_{\psi} \sim n$ as $n\to\infty$. For the self-energy function Σ_{ψ} an expansion in inverse powers of n follows, with the leading contribution being of order n. Consequently $G_{\psi} = \mathrm{O}(1/n)$, and since $G_{\phi} = \mathrm{O}(1)$, only a finite number of graphs contribute to Σ_{ϕ} and Σ_{ψ} in calculations up to a given order in $1/n$. If we restrict ourselves to the first nontrivial order $1/n$, we may drop all contributions represented by the ellipsis in these two equations.

According to equations (16), the free parts $\tilde{G}_\psi^{(0)-1}$ and $\tilde{G}_\phi^{(0)-1}$ are analytic in the momenta p and q. Since the asymptotic forms of the self-consistent solutions involve nontrivial powers of momenta for generic d, as we shall see, these free parts do not contribute to $\tilde{G}_{\phi,\psi}^{(\mathrm{as})}(p,q)$ and may be discarded. From equation (21) we thus find

$$\tilde{G}_\psi^{(\mathrm{as})}(p,q) = \frac{2}{n\,\lambda\,F(p,q)}, \tag{22}$$

while equation (20) yields

$$\left[\tilde{G}_\phi^{(\mathrm{as})}(p,q)\right]^{-1} = \frac{2}{n}\int_{\boldsymbol{p}'}^{(\bar{m})}\int_{\boldsymbol{q}'}^{(m)} \frac{\tilde{G}_\phi^{(\mathrm{as})}(|\boldsymbol{p}'+\boldsymbol{p}|,|\boldsymbol{q}'+\boldsymbol{q}|)}{F(p',q')} \tag{23}$$

with

$$F(p,q) = \int_{\boldsymbol{p}'}^{(\bar{m})}\int_{\boldsymbol{q}'}^{(m)} \tilde{G}_\phi^{(\mathrm{as})}(|\boldsymbol{p}'+\boldsymbol{p}|,|\boldsymbol{q}'+\boldsymbol{q}|)\,\tilde{G}_\phi^{(\mathrm{as})}(p',q'). \tag{24}$$

Just like $\tilde{G}_{\phi,\psi}^{(\mathrm{as})}(p,q)$, $F(p,q)$ is a generalized homogeneous function; equation (24) can be combined with the scaling form (14) of $\tilde{G}_\phi^{(\mathrm{as})}$ for concluding that

$$F(p,q) = p^{\bar{m}+\theta m-4\tilde{\Delta}_\phi}\,F(1,p^{-\theta}q) \tag{25}$$

$$= q^{(\bar{m}+\theta m-4\tilde{\Delta}_\phi)/\theta}\,F(q^{-1/\theta}p,1). \tag{26}$$

We can now insert these expressions together with their analogues (14) for $\tilde{G}_{\phi,\psi}^{(\mathrm{as})}(p,q)$ into equation (23), choosing either one of the components $\boldsymbol{p}$ and $\boldsymbol{q}$ of the external momentum to be zero. Matching the amplitudes of the corresponding powers $p^{2\tilde{\Delta}_\phi}$ and $q^{2\tilde{\Delta}_\phi/\theta}$ on both sides of the resulting two equations yields the consistency conditions

$$\frac{n}{2} = \int_{\boldsymbol{p}}^{(\bar{m})} \frac{p^{4\tilde{\Delta}_\phi-\bar{m}}}{|\boldsymbol{p}+\mathbf{1}|^{2\tilde{\Delta}_\phi}}\int_{\boldsymbol{q}}^{(m)} \frac{\tilde{G}_\phi^{(\mathrm{as})}\big(1,p^\theta|\boldsymbol{p}+\mathbf{1}|^{-\theta}q\big)}{F(1,q)} \equiv I_1(n) \tag{27}$$

and

$$\frac{n}{2} = \int_{\boldsymbol{q}}^{(m)} \frac{q^{4\tilde{\Delta}_\phi/\theta-m}}{|\boldsymbol{q}+\mathbf{1}|^{2\tilde{\Delta}_\phi/\theta}}\int_{\boldsymbol{p}}^{(\bar{m})} \frac{\tilde{G}_\phi^{(\mathrm{as})}\big(q^{1/\theta}\,|\boldsymbol{q}+\mathbf{1}|^{-1/\theta}\,p,1\big)}{F(p,1)} \equiv I_2(n), \tag{28}$$

where $\mathbf{1}$ designates an arbitrary unit vector in $\bar{m}$ or m dimensions.

Since the left-hand sides are of order n, so must be the right-hand sides. Hence the integrals $I_1(n)$ and $I_2(n)$ must have simple poles at $1/n = 0$. Consider first $I_1(n)$. Its inner integral $\int_{\boldsymbol{q}}^{(m)}$ approaches a $\boldsymbol{p}$-independent value as $p\to\infty$. Recalling equations (15) and (18), we see that the ratio of $\boldsymbol{p}$-dependent factors in front of this integral behaves as $\sim p^{2-\bar{m}+\mathrm{O}(1/n)}$ in this limit. Had we regularized the $\boldsymbol{p}$ integral with a large-$\boldsymbol{p}$ cut-off Λ, it would have had ultraviolet (uv) divergences of the form $\Lambda^{2-s+\mathrm{O}(1/n)}$, $s = 0, 1, 2$. In the dimensionally regularized theory, uv divergences $\sim\Lambda^{\mathrm{O}(1/n)} = 1 + \mathrm{O}(1/n)\ln\Lambda + \cdots$ become poles at $1/n = 0$. Conversely, contributions from the inner integral $\int_{\boldsymbol{q}}^{(m)}$ falling off faster than p^{-2} as $n\to\infty$ do not contribute to the residue of the pole $\sim(1/n)^{-1}$, and hence may be dropped when calculating it.

The upshot is that the $\mathrm{O}(n)$ contribution of $I_1(n)$ can be determined as follows. We add and subtract from $\tilde{G}_\phi^{(\mathrm{as})}(1,p^\theta|\boldsymbol{p}+\mathbf{1}|^{-\theta}q)$ its Taylor expansion in the second variable $u_p(q) \equiv p^\theta|\boldsymbol{p}+\mathbf{1}|^{-\theta}q$ about its limiting value for $p\to\infty$, $u_\infty(q) = q$, to the order necessary to ensure that no contributions of order n are produced by the difference. Details of this calculation are described in appendix A. The result is

$$I_1(n) = n\,\frac{K_{\bar{m}}}{\eta_{\mathrm{L2}}^{(1)}\,\bar{m}}\int_{\boldsymbol{q}}^{(m)} \frac{\mathcal{P}_1(q^4)}{(1+q^4)^3\,I(1,q)} + \mathrm{O}(n^0). \tag{29}$$

Here $K_{\bar m}$ is a standard geometric factor defined by

$$K_D \equiv (4\pi)^{-D/2}\,\frac{2}{\Gamma(D/2)}, \tag{30}$$

while $\mathcal{P}_1$ means the polynomial

$$\mathcal{P}_1(q^4) = 4 - \bar m\,(1+q^4). \tag{31}$$

The integral $I(1,q)$, defined by

$$I(p,q) = \int_{p'}^{(\bar m)}\int_{q'}^{(m)} \frac{1}{(p'^2+q'^4)\left(|\boldsymbol{p}'+\boldsymbol{p}|^2+|\boldsymbol{q}'+\boldsymbol{q}|^4\right)}, \tag{32}$$

is the counterpart of Ma's (1973) critical 'elementary bubble' $\Pi(k)$ for the present case of m-axial LPs.

The function $I(p,q)$ satisfies scaling relations similar to those of $\tilde G^{\rm as}_{\phi,\psi}(p,q)$ in equation (14):

$$I(p,q) = p^{-\epsilon}\,I\left(1,qp^{-1/2}\right) = q^{-2\epsilon}\,I\left(pq^{-2},1\right). \tag{33}$$

Further properties of this function are discussed in appendix B.

The O(n) term of $I_2(n)$ can be worked out in a similar fashion. To calculate it, we must add and subtract the Taylor expansion of the corresponding inner integral to fourth order in one of its variables (see appendix A). Proceeding in this manner yields

$$I_2(n) = \frac{n}{2}\,\frac{K_m}{\left(\eta^{(1)}_{\rm L2}+4\theta^{(1)}\right)m(m+2)}\int_p^{(\bar m)}\frac{\mathcal{P}_2(p^2)}{(1+p^2)^5\,I(p,1)} + {\rm O}(n^0) \tag{34}$$

with the polynomial

$$\mathcal{P}_2(p^2) = 3(8-m)(6-m) + 5(m^2+2m-96)\,p^2 + (m^2+50m+144)\,p^4 - m(m+2)\,p^6. \tag{35}$$

Using the expressions (29) and (34) for the integrals $I_1(n)$ and $I_2(n)$, we now solve the equations (27) and (28) for $\eta^{(1)}_{\rm L2}$ and $\theta^{(1)}$. This gives

$$\eta_{\rm L2} = \frac{1}{n}\,\frac{2K_{\bar m}}{\bar m}\int_q^{(m)}\frac{\mathcal{P}_1(q^4)}{(1+q^4)^3}\,\frac{1}{I(1,q)} + {\rm O}(n^{-2}), \tag{36}$$

$$\theta = \frac{1}{2} - \frac{\eta_{\rm L2}}{4} + \frac{1}{n}\,\frac{K_m}{4m(m+2)}\int_p^{(\bar m)}\frac{\mathcal{P}_2(p^2)}{(1+p^2)^5}\,\frac{1}{I(p,1)} + {\rm O}(n^{-2}). \tag{37}$$

The implied $1/n$ expansion of $\eta_{\rm L4}$ follows with the aid of relation (19) for the $1/n$ coefficient $\eta^{(1)}_{\rm L4}$; it reads

$$\eta_{\rm L4} = \frac{1}{n}\,\frac{2K_m}{m(m+2)}\int_p^{(\bar m)}\frac{\mathcal{P}_2(p^2)}{(1+p^2)^5}\,\frac{1}{I(p,1)} + {\rm O}(n^{-2}). \tag{38}$$

In equations (37) and (38) we have expressed the $1/n$ coefficients in terms of an integral over the perpendicular momentum $\boldsymbol{p}$. Upon exploiting the scaling property (33) of the integral $I(p,q)$, we could rewrite this as an integral over the parallel momentum $\boldsymbol{q}$. The transformed integrand would contain the polynomial $\tilde{\mathcal{P}}_2(q^4) = q^{12}\mathcal{P}_2(q^{-4})$. Likewise, the $\boldsymbol{q}$ integral in equation (36) can be recast as an integral over $\boldsymbol{p}$.

The above equations (36)–(38), giving the expansions of the correlation exponents $\eta_{\rm L2}$, $\eta_{\rm L4}$ and the anisotropy exponent θ to order $1/n$ for general values of (m,d) with $2+m/2 < d < 4+m/2$, are the main results of this paper. In the following we further analyse these results for special choices of (m,d). Figure 1 illustrates the region between the upper and lower critical lines in which nonclassical behaviour is expected and displays the special cases to be discussed below as full circles.

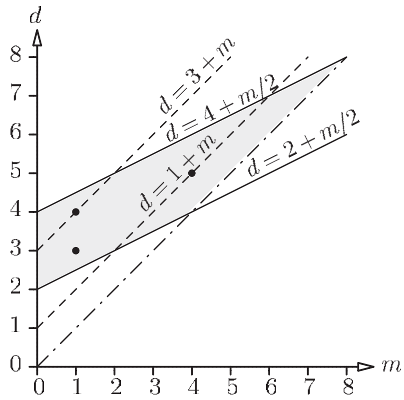


Figure 1. Lines of upper and lower critical dimensions $d^*(m) = 4 + m/2$ and $d_\ell(m) = 2 + m/2$, respectively. In the shaded region, bounded by portions of these lines together with the condition $0 \leqslant m \leqslant d$, nonclassical critical behaviour is expected to occur. The full circles indicate the special cases discussed in section 7. Also shown are the lines $d = 3 + m$ and $d = m + 1$ on which the free propagator simplifies to a simple exponential and incomplete gamma function, respectively (see section 7).

4. The limiting cases of critical points and isotropic Lifshitz points

In this section we check that our results for general m, equations (36)–(38), comply with known series expansions to $O(1/n)$ for the cases $m = 0$ of CPs (Ma 1973, Abe and Hikami 1973) and $m = d$ of isotropic LPs (Hornreich *et al* 1975b). In either case we must take limits $D \to 0$ of D-dimensional integrals to determine the $O(1/n)$ coefficients, where $D = m$ or $\bar{m}$. To do this, we use the relation

$$\lim_{D\to 0} \int \mathrm{d}^D x\, g(x; D) = \lim_{D\to 0} S_D \int \mathrm{d}x\, x^{D-1} g(x; D) = g(0; 0). \tag{39}$$

Consider first the CP case $m \to 0$. Here only the correlation exponent η_{L2} is physically meaningful. From equation (36) one easily derives

$$\begin{aligned} \lim_{m\to 0} \eta_{L2} &= \frac{2}{d} K_d \lim_{m\to 0} \frac{\mathcal{P}_1(0)}{I(1,0)} \frac{1}{n} + O(1/n^2) \\ &= \frac{2}{d} K_d \frac{4-d}{J_d(1,1)} \frac{1}{n} + O(1/n^2), \end{aligned} \tag{40}$$

where $J_d(1, 1)$ is a standard one-loop integral, corresponding to a special case of the quantity $J_D(f, t)$ defined by equation (A.1) of appendix A. Its explicit value is given in equation (A.9). Inserting it into the last equation we immediately recover the familiar result for η first derived by Ma (1973) and Abe and Hikami (1973):

$$\eta \equiv \eta_{L2}|_{m=0} = \frac{4(4-d)\,\Gamma(d-2)}{d\,\Gamma(2-d/2)\,\Gamma^2(d/2-1)\,\Gamma(d/2)} \frac{1}{n} + O(1/n^2). \tag{41}$$

The limit $m \to d$, corresponding to an isotropic LP, can be handled in a similar fashion. Now only the correlation exponent η_{L4} is physically significant. Equation (38) yields

$$\begin{aligned} \lim_{m\to d} \eta_{L4} &= \frac{2K_d}{d(d+2)} \lim_{m\to d} \frac{\mathcal{P}_2(0)}{I(0,1)} \frac{1}{n} + O(1/n^2) \\ &= 6\, \frac{(d-8)(d-6)}{d(d+2)} \frac{K_d}{J_d(2,2)} \frac{1}{n} + O(1/n^2) \end{aligned} \tag{42}$$

with $J_d(2,2)$ given by equation (A.10). Upon making several transformations of Gamma functions, we recover the result of Hornreich *et al* (1975b) for η_{L4}, namely

$$\eta_{L4}|_{m=d} = 3(8-d)\,2^{d-2}\,\frac{\sin(\pi d/2)\,\Gamma[(d-3)/2]}{\pi^{3/2}\,d(d+2)\,\Gamma(d/2)}\,\frac{1}{n} + \mathrm{O}(1/n^2). \tag{43}$$

5. Consistency with ϵ expansion about the upper critical dimension

In Diehl and Shpot (2000) and Shpot and Diehl (2001) the ϵ expansions about the upper critical dimension $d^* = 4+m/2$ of all critical, crossover and the usual correction-to-scaling exponents have been obtained for general m and n. These series can be expanded in powers of $1/n$ to $\mathrm{O}(1/n)$. Conversely, considering the limit of small ϵ, our above $\mathrm{O}(1/n)$ results for general (m,d) can be expanded in powers of ϵ. The resulting two double-series expansions in ϵ and $1/n$ of each exponent should agree.

We shall work out the $\mathrm{O}(\epsilon^2/n)$ contributions implied by the $1/n$ expansions (36) and (38) for general m, expressing them in terms of multi-dimensional integrals. Unfortunately, these integrals are in general rather complicated and not necessarily analytically tractable. For this reason, we will content ourselves here with verifying the consistency of the series expansions of η_{L2} and η_{L4} in ϵ and $1/n$ for the special choice $m=2$. Owing to the simple form which the scaling functions of the position-space propagators take in this case both for $n=\infty$ and the Gaussian theory (simple exponentials, see Mergulhão and Carneiro (1999), Diehl and Shpot (2000), Shpot and Diehl (2001)), the $\mathrm{O}(\epsilon^2/n)$ terms of the series expansions can be worked out analytically in a straightforward fashion (see below).

One source of ϵ dependence in equations (36) and (38) is the function $I(p,q)$. This is just the one-loop Feynman integral associated with the four-point graph  of the usual ϕ^4 theory for m-axial LPs. Its Laurent expansion in ϵ reads

$$I(p,q) = \frac{p^{-\epsilon}}{\epsilon}c_{-1}\left[1-\epsilon\,F(q^2/p)\right] + \mathrm{O}(\epsilon), \tag{44}$$

where c_{-1} and minus the scaling function F represent the residuum and finite part of $I(1,q)$, respectively.

In the case of η_{L2}, a second source of ϵ dependence is the function $\mathcal{P}_1(q^4)$ (see equation (31)). Upon decomposing it as

$$\mathcal{P}_1(q^4) = \mathcal{P}_1^{(0)}(q^4) + \epsilon\,\mathcal{P}_1^{(1)}(q^4) \tag{45}$$

with

$$\mathcal{P}_1^{(0)}(q^4) = \frac{m}{2}(1+q^4) - 4q^4 \tag{46}$$

and

$$\mathcal{P}_1^{(1)}(q^4) = 1 + q^4, \tag{47}$$

we arrive at the ϵ expansion

$$\eta_{L2} = \frac{1}{n}\frac{2K_{\bar m}}{\bar m}\frac{\epsilon}{c_{-1}}\int_q^{(m)}\frac{\mathcal{P}_1^{(0)}(q^4)+\epsilon\,\mathcal{P}_1^{(1)}(q^4)}{(1+q^4)^3}\left[1+\epsilon\,F(q^2)\right] + \mathrm{O}\left(\frac{\epsilon^3}{n},\frac{\epsilon^2}{n^2}\right). \tag{48}$$

The result suggests the presence of a term linear in ϵ. However, the ϵ expansions of the correlation exponents η_{L2} and η_{L4} are known to start at order ϵ^2, so this term must vanish. It does indeed, since

$$\int_0^\infty \mathrm{d}q\,q^{m-1}\frac{\mathcal{P}_1^{(0)}(q^4)}{(1+q^4)^3} = 0. \tag{49}$$

Hence we have

$$\eta_{L2} = \frac{\epsilon^2}{n}\frac{4K_{4-m/2}}{(8-m)\,c_{-1}}\int_q^{(m)}\frac{\mathcal{P}_1^{(0)}(q^4)\,F(q^2)+\mathcal{P}_1^{(1)}(q^4)}{(1+q^4)^3}+\mathrm{O}\left(\frac{\epsilon^3}{n},\frac{\epsilon^2}{n^2}\right). \tag{50}$$

Turning to the expansion (38) of η_{L4}, we note that the polynomial $\mathcal{P}_2(p^2)$ in its integrand does not depend on ϵ. However, the integration measure contains a factor $p^{-\epsilon}$ which cancels the one originating from $I(p,1)$ (see equation (44)). It follows that

$$\eta_{L4} = \frac{1}{n}\frac{2K_m}{m(m+2)}\frac{\epsilon}{c_{-1}}\int_p^{(\bar{m}^*)}\frac{\mathcal{P}_2(p^2)}{(1+p^2)^5}\,[1+\epsilon\,F(1/p)]+\mathrm{O}\left(\frac{\epsilon^3}{n},\frac{\epsilon^2}{n^2}\right). \tag{51}$$

Similarly to above, the contribution from the 1 in the square brackets vanishes because

$$\int_0^\infty \mathrm{d}p\,p^{3-m/2}\frac{\mathcal{P}_2(p^2)}{(1+p^2)^5} = 0 \tag{52}$$

for general m. Thus the contribution linear in ϵ is zero, and the ϵ expansion of the large-n result (38) becomes

$$\eta_{L4} = \frac{\epsilon^2}{n}\frac{2K_m}{m(m+2)\,c_{-1}}\int_p^{(\bar{m}^*)}\frac{\mathcal{P}_2(p^2)\,F(1/p)}{(1+p^2)^5}+\mathrm{O}\left(\frac{\epsilon^3}{n},\frac{\epsilon^2}{n^2}\right). \tag{53}$$

Equations (50) and (53) give the contributions of $\mathrm{O}(\epsilon^2/n)$ implied by our large-n results for general m. They should agree with those obtained from the ϵ-expansion results of Diehl and Shpot (2000) and Shpot and Diehl (2001). Let us verify this explicitly for the biaxial case $m = 2$, for which both the residue c_{-1} and the function F appearing in the Laurent expansion (44) may be gleaned from Mergulhão and Carneiro (1999). Taking into account that the expansion parameter $\epsilon_\parallel$, called ϵ by these authors, corresponds to $\epsilon_\parallel = 2\epsilon$ in our notation[7], one sees that

$$c_{-1}|_{m=2} = \frac{1}{32\pi^2}, \tag{54}$$

$$F(q^2)|_{m=2} = \frac{q^2}{2}\arctan(2/q^2)+\frac{1}{2}\ln(1+q^4/4)-1-\ln 4. \tag{55}$$

Upon substituting this into equations (50) and (53), one can perform the required integrations to obtain

$$\eta_{L2}(m=2) = \frac{4}{9}\frac{\epsilon^2}{n}+\mathrm{O}\left(\frac{\epsilon^3}{n},\frac{\epsilon^2}{n^2}\right), \tag{56}$$

$$\eta_{L4}(m=2) = -\frac{8}{27}\frac{\epsilon^2}{n}+\mathrm{O}\left(\frac{\epsilon^3}{n},\frac{\epsilon^2}{n^2}\right). \tag{57}$$

Expanding the correct $\mathrm{O}(\epsilon^2)$ results (Sak and Grest 1978, Mergulhão and Carneiro 1999, Shpot and Diehl 2001) to order $1/n$ yields identical results.

Since the Laurent expansion of the integral $I(p,q)$ to order ϵ^0 is also explicitly known when $m = 6$ (where $d^* = 7$) (Mergulhão and Carneiro 1999), a similar consistency check should also be possible in this case via analytical, albeit somewhat more complicated, calculations.

A complete proof of agreement of the results of the $1/n$ and ϵ expansions for general m lies beyond the scope of the present work. The main reason that we have so far been unable to generalize the foregoing consistency check to general values of m is our lack of knowledge of a sufficiently simple closed form for the finite part of $I(p,q)$.

[7] Note that there is a misprint in the corresponding formula of Shpot and Diehl (2001) which precedes its equation (69). The correct relation between $\epsilon_\parallel$ and $\epsilon = d^*(m) - d$ is as given above in the main text.

6. Consistency with expansions about the lower critical dimension

The $1/n$ expansions (36) and (38) should hold down to the lower critical dimensionality d_ℓ, which is given by the line $d_\ell(m) = 2 + m/2$ in our case (unless the LP gets destroyed by fluctuations). Considering dimensions $d = d_\ell + \epsilon_\ell$ slightly above d_ℓ, we can additionally expand in ϵ_ℓ. Conversely, any pertinent series expansion in ϵ_ℓ that is available can be expanded in powers of $1/n$. The resulting pairs of double power series in $1/n$ and ϵ must agree.

Series expansions about the lower critical dimension can be obtained for systems of the kind we are concerned with here—namely, systems having a low-temperature phase with spontaneously broken O(n) symmetry—by analysing appropriate nonlinear sigma models (Polyakov 1975, Brézin and Zinn-Justin 1976a, 1976b, Bardeen *et al* 1976). There are two sorts of results we can compare with: ϵ_ℓ expansions for the CP case $m = 0$ (Brézin and Zinn-Justin 1976b), on the one hand, and ones for m-axial LPs (Grest and Sak 1978), on the other hand. By investigating an appropriate generalization of the conventional nonlinear sigma model, the latter authors produced one-loop results for the three special values $m = 1, 2$ and 4.

To show that our O($1/n$) results (36)–(38) are compatible with these ϵ_ℓ expansions, we proceed in much the same way as in section 5. The integrals $I(p, 1)$ and $I(1, q)$ in the denominators of these equations have poles at $\epsilon_\ell = 0$. According to equations (B.8) and (B.23),

$$I(p,1) = I(1,q) + \mathrm{O}(\epsilon_\ell^0) = (4\pi)^{-1-m/4}\,\frac{\Gamma(m/4)}{\Gamma(m/2)}\,\frac{2}{\epsilon_\ell} + \mathrm{O}(\epsilon_\ell^0). \tag{58}$$

Substituting this into equations (36)–(38), one can perform the remaining integrations. Unlike in the case of small ϵ, nonzero contributions linear in ϵ_ℓ appear. We obtain

$$\eta_{\mathrm{L2}} = \frac{\eta_{\mathrm{L4}}}{2} + \mathrm{O}\left(\frac{\epsilon_\ell^2}{n}, \frac{\epsilon_\ell}{n^2}\right) = \frac{\epsilon_\ell}{n} + \mathrm{O}\left(\frac{\epsilon_\ell^2}{n}, \frac{\epsilon_\ell}{n^2}\right) \tag{59}$$

and

$$\theta = \frac{1}{2} + \mathrm{O}\left(\frac{\epsilon_\ell^2}{n}, \frac{\epsilon_\ell}{n^2}\right). \tag{60}$$

These results are consistent both with Grest and Sak (1978) as well as with Brézin and Zinn-Justin (1976b). Since they were derived for general m, the above critical exponents are *independent* of m to the indicated order in $1/n$ and ϵ. Note that the O(ϵ_ℓ/n) correction to the anisotropy exponent vanishes.

From the structure of the ϵ_ℓ expansions it is clear that the O(ϵ_ℓ) terms of η_{L2} and η_{L4} must be inversely proportional to $n - 2$. Hence we can generalize Grest and Sak's (1978) results for $m = 1, 2$ and 4 to conclude that

$$\eta_{\mathrm{L2}}(m) = \frac{1}{2}\eta_{\mathrm{L4}}(m) + \mathrm{O}(\epsilon_\ell^2) = \frac{\epsilon_\ell}{n-2} + \mathrm{O}(\epsilon_\ell^2), \tag{61}$$

$$\theta(m) = \tfrac{1}{2} + \mathrm{O}(\epsilon_\ell^2), \tag{62}$$

for general m.

As one sees, the series coefficients of the terms linear in ϵ_ℓ are independent of m. This is a feature they have in common with the coefficients of the O(ϵ) contributions of *all* bulk critical exponents of the m-axial LPs we are concerned with here (Shpot and Diehl 2001). For the ϵ expansion of the above exponents this is trivially fulfilled since their O(ϵ) terms are zero. However, other exponents such as those associated with the parallel and perpendicular correlation lengths do have O(ϵ) contributions, whose coefficients do not depend on m.

7. Special cases

In this section we further exploit our results for general m, equations (36)–(38), by considering special cases. Of primary physical interest clearly is the case $(m, d) = (1, 3)$ of a three-dimensional system with uniaxial anisotropy. Unfortunately, this case appears too difficult to allow a completely analytic calculation of the $O(1/n)$ terms. Let us therefore first consider some simpler cases, before returning to it.

7.1. *The case $m = 1, d = 4$*

For this choice of (m, d), the required calculations can be done analytically to obtain closed expressions for the $O(1/n)$ coefficients of the exponents η_{L2}, η_{L4} and θ.

To see this, recall that the free propagator in position space takes a simple exponential form on the whole line $d = 3 + m$ (Mergulhão and Carneiro 1999, Shpot and Diehl 2001). On it, the scaling function $\Phi_{m,d}(\upsilon)$ of the Fourier back transform

$$G_\phi^{(0)}(r, z) = r^{-2+\epsilon}\,\Phi_{m,d}(\upsilon), \qquad \upsilon \equiv z/\sqrt{r}, \tag{63}$$

of the free momentum-space propagator $\tilde{G}_\phi^{(0)}(p, q)$ introduced in equation (16) simplifies to

$$\Phi_{m,3+m}(\upsilon) = (4\pi)^{-2+\epsilon}\mathrm{e}^{-\upsilon^2/4}, \tag{64}$$

whilst away from it, it is generally a difference of two generalized hypergeometric functions, a so-called Fox–Wright ${}_1\Psi_1$ function (Shpot and Diehl 2001).

Previously, this simplifying feature was exploited in the context of the ϵ expansion in two different ways: Mergulhão and Carneiro (1999) fixed the codimension $\bar{m} = d - m$ at $\bar{m} = 3$ and took $m = 2 - 2\epsilon$ (and hence $d = 5 - 2\epsilon$) to expand about the point $(m, d) = (2, 5)$. Shpot and Diehl (2001) performed a two-loop RG analysis for general fixed m and dimensions $d = d^*(m) - \epsilon$. Owing to the simple form (64), both kinds of calculations could be performed analytically for $m = 2$, as well as for $m = 6$ where one can benefit from similar simplifications at $d^*(6) = 7$. Despite the difference of the two procedures, they yielded consistent results for the ϵ expansions of the critical exponents about $(m, d) = (2, 5)$ to $O(\epsilon^2)$.[8]

Since the ϵ expansion is asymptotic, it gives us direct information only about the behaviour in the immediate vicinity of the line of upper critical dimensions $d^*(m)$. In order to derive from it reliable predictions for the values of critical exponents in $d = 3$ dimensions, one must extrapolate, for example, in the case $m = 1$ of a uniaxial LP down to $\epsilon = 3/2$. This is not normally possible unless sophisticated extrapolation and resummation methods are employed. The $1/n$ expansion, on the other hand, does not require ϵ to be small and hence enables us to move further away from d^*. Here we exploit the simple form (64) to gain information about the behaviour at $(m, d) = (1, 4)$, corresponding to a distance of $\epsilon = 1/2$ from $d^*(1) = 9/2$.

It is no complicated matter to compute the function $I(1, q)$ for $m = 1$ and $d = 4$; one obtains

$$I(1, q)|_{m=1}^{d=4} = \frac{1}{8\pi\sqrt{2}}\sqrt{\sqrt{q^4+4} - q^2} = \frac{1}{4\pi\sqrt{2}}\frac{1}{\sqrt{\sqrt{q^4+4} + q^2}}, \tag{65}$$

which yields $I(p, 1) = p^{-1/2}\,I(1, p^{-1/2})$ by virtue of the scaling property (33). Upon inserting these expressions into equations (36)–(38), one can perform the required integrations

[8] Verifying the consistency of Mergulhão and Carneiro's (1999) series expansions to $O(\epsilon^2)$ about $(m, d) = (2, 5)$ and $(6, 7)$ with those of Shpot and Diehl (2001) requires the correction of two minor mistakes in the former reference; see section 4.2 of Shpot and Diehl (2001) for details.

in a straightforward manner. This gives the following results for the $1/n$ coefficients of the exponents η_{L2}, θ and η_{L4}:

$$\left.\begin{aligned} \eta_{L2}^{(1)} &= \tfrac{5}{9\pi\sqrt{3}} \simeq 0.1021, \\ \theta^{(1)} &= -\tfrac{4}{27\pi\sqrt{3}} \simeq -0.0272, \\ \eta_{L4}^{(1)} &= -\tfrac{2}{27\pi\sqrt{3}} \simeq -0.0136, \end{aligned}\right\} \qquad m = 1, \quad d = 4. \tag{66}$$

Despite the smallness of the $O(1/n)$ corrections, these results provide clear evidence for the fact that the critical exponents η_{L2}, η_{L4} and θ have nonclassical values below the upper critical dimension. This conclusion is in full accord with previous work based on the ϵ expansion (Diehl and Shpot 2000, Shpot and Diehl 2001).

7.2. *The special case $m = 4, d = 5$*

There is another line on which the scaling function $\Phi_{m,d}$ simplifies: for $d = m + 1$ it reduces to an incomplete gamma function (see equation (18) of Shpot and Diehl (2001)). Specifically for $d = m + 1 = 5$, it becomes the elementary function

$$\Phi_{4,5}(\upsilon) = \frac{1}{2(2\pi)^2}\,\frac{1}{\upsilon^2}\left(1 - e^{-\upsilon^2/4}\right). \tag{67}$$

Its simplicity enables us to compute the integral $I(1, q)$ for this choice of m and d without much difficulty. We obtain

$$I(1, q)|_{m=4}^{d=5} = \frac{1}{2(2\pi)^2}\,\frac{1}{4q^2}\left\{q^2 \arctan\left[\frac{2}{q^2(q^4+3)}\right] + \ln\frac{1+q^4}{1+q^4/4}\right\} \tag{68}$$

and a corresponding result for $I(p, 1) = p^{-1} I(1, p^{-1/2})$. These expressions can be substituted into equations (36)–(38) and the required integrals then evaluated by numerical means. This yields the values

$$\left.\begin{aligned} \eta_{L2}^{(1)} &\simeq 0.314, \\ \theta^{(1)} &\simeq -0.0728, \\ \eta_{L4}^{(1)} &\simeq 0.045, \end{aligned}\right\} \qquad m = 4, \quad d = 5, \tag{69}$$

for the $1/n$ coefficients of η_{L2}, θ and η_{L4}. Remarkably, the $1/n$ coefficient of η_{L4} is no longer negative, as it was both for infinitesimally small ϵ and at $(m, d) = (1, 4)$.

7.3. *The case $m = 1, d = 3$*

We now turn to the uniaxial, three-dimensional case $(m, d) = (1, 3)$. Unfortunately, we have not been able to evaluate fully analytically the required integrals of the $1/n$ coefficients.

Let us start from equation (32). The $(\bar{m} = 2)$-dimensional integration over the perpendicular momentum $\boldsymbol{p}'$ can be performed in a straightforward fashion, giving

$$I(1, q) = \frac{1}{8\pi^2}\int_{-\infty}^{\infty} dq'\, A^{-1/2} \ln\left[\frac{2q'^4(q'+q)^4 + A + \left[q'^4 + (q'+q)^4 + 1\right]A^{1/2}}{2q'^4(q'+q)^4}\right], \tag{70}$$

where A stands for the expression

$$A(q', q) = \left[1 + (q'+q)^4\right]^2 + (1 + q'^4)^2 - 2q'^4(q'+q)^4 - 1. \tag{71}$$

The integration over q' can be regarded as an integration in the complex q' plane and after some work be shown to reduce to[9]

$$I(1, q) = -\frac{i}{2\pi}\int_0^{\frac{i}{2q}} \frac{dq'}{\sqrt{(1 + 4q'^2q^2)\left[1 + \frac{1}{4}(4q'^2 + q^2)^2\right]}}. \tag{72}$$

[9] We are indebted to S Rutkevich for this calculation.

This in turn can be expressed as a complete elliptic integral or Gauss hypergeometric function to obtain

$$I(1,q) = \tfrac{1}{4}(4+q^4)^{-1/4}(1+q^4)^{-1/2}\,{}_2F_1\big(\tfrac{1}{2},\tfrac{1}{2};1;k^2\big) \qquad (73)$$

with

$$k^2 = \frac{1}{2}\left[1 - \frac{3+q^4}{(1+q^4)(1+4/q^4)^{1/2}}\right]. \qquad (74)$$

Known quadratic and linear transformation formulae for the hypergeometric function ${}_2F_1$, such as equations (15.30.30) and (15.3.4) of Abramowitz and Stegun (1972), enable us to cast the above integral in the following two equivalent forms:

$$I(1,q) = \frac{1}{4\sqrt{2}}\,u^{1/4}\,{}_2F_1\big(\tfrac{1}{4},\tfrac{1}{4};1;u\big), \qquad u = \frac{4}{(1+q^4)^2(4+q^4)}, \qquad (75)$$

$$= \frac{1}{4\sqrt{2}}\,w^{1/4}\,{}_2F_1\big(\tfrac{1}{4},\tfrac{3}{4};1;-w\big), \qquad w = \frac{4}{q^4(3+q^4)^2}. \qquad (76)$$

Upon substitution of either one of them into our general expressions (36)–(38) for the $O(1/n)$ coefficients (along with their counterparts for $I(p,1) = p^{-3/2}I(1,p^{-1/2})$), the remaining integration over q can easily be performed numerically. The results are

$$\eta_{L2}(m=1,d=3) \simeq 0.306\,\frac{1}{n} + O(n^{-2}), \qquad (77)$$

$$\theta(m=1,d=3) \simeq \frac{1}{2} - 0.0487\,\frac{1}{n} + O(n^{-2}), \qquad (78)$$

$$\eta_{L4}(m=1,d=3) \simeq 0.223\,\frac{1}{n} + O(n^{-2}). \qquad (79)$$

Just as in the previous case of $m = 4$, $d = 5$, and unlike the coefficient of the $O(\epsilon^2/n)$ term, $\eta_{L4}^{(1)}$ is *positive*. This suggests a tendency of η_{L4} to change from small negative values for $d \lesssim d^*$ to positive ones as d is lowered further.

8. A universal scaling function and related amplitude

So far we have focused our attention on the calculation of critical exponents. However, the $1/n$ expansion can also be employed to compute other universal quantities such as universal amplitude ratios and scaling functions. As an example, we here compute the leading nontrivial term of a universal scaling function associated with the energy-density cumulant at the LP and a related amplitude.

According to general scaling arguments and RG work (Diehl and Shpot 2000, Shpot and Diehl 2001, Diehl *et al* 2003a, 2003b), the leading singular part of this function is expected to take on large length scales the asymptotic scaling form

$$\tilde{G}^{(\mathrm{sing})}_{\phi^2}(p,q) \equiv \langle\phi^2\phi^2\rangle^{\mathrm{cum}}_{p,q}|_{\mathrm{sing}} \approx E_1\,p^{-\alpha_L/\nu_{L2}}\,\Psi_{m,d}(E_2\,qp^{-\theta}). \qquad (80)$$

Here the quantity on the left-hand side is the Fourier transform of the cumulant $\langle\phi^2(\boldsymbol{x})\phi^2(\boldsymbol{0})\rangle^{\mathrm{cum}}$, where 'sing' means singular part. Further, α_L and ν_{L2} are the usual critical indices of the specific heat and perpendicular correlation length; they are related to the exponents used above via

$$\alpha_L/\nu_{L2} = 2/\nu_{L2} - \bar{m} - m\theta = 2\tilde{\Delta}_\psi. \qquad (81)$$

Furthermore, E_1 and E_2 denote nonuniversal metric factors. The former can be varied by changing the normalization of the order parameter field ϕ, the latter by modifying the scale in

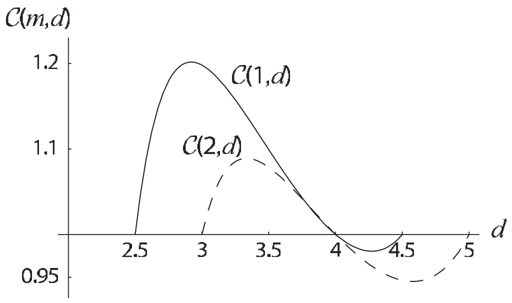


Figure 2. Universal amplitude $\mathcal{C}(m, d)$ as a function of d for $m = 1$ (full line) and $m = 2$ (broken line).

which parallel momenta are measured, i.e. by multiplying the term $(\partial_z^2\phi)^2$ of the Hamiltonian by a factor $\sigma \neq 1$.

The scaling function $\Psi_{m,d}$ should be universal, but depends of course on n. We choose its normalization such that $\Psi_{m,d}(0) = 1$. To fix the scale of its argument, we could require that the logarithmic derivative of $\Psi_{m,d}(\mathsf{q})$ at a reference value q_0 (for instance, $\mathsf{q}_0 = 0$) takes a certain value. We do this simply by making the choice $E_2 = 1$. The $1/n$ expansion of $\Psi_{m,d}$ then starts at order $(1/n)^0$. Taking into account that the function $\tilde{G}_{\phi^2}(p, q)$ is trivially related to $\tilde{G}_\psi(p, q)$ (see e.g. chapter 29 of Zinn-Justin (1996)) and recalling equation (21), one sees that

$$\Psi_{m,d}(\mathsf{q}) = \frac{I(1, \mathsf{q})}{I(1, 0)} + \mathrm{O}(1/n). \tag{82}$$

If we take the limit $p \to 0$ at fixed q, the scaling form (80) must yield a p-independent contribution $\sim q^{-\alpha_{\mathrm{L}}/\nu_{\mathrm{L}2}\theta}$. Hence the scaling function must have the asymptotic behaviour

$$\Psi_{m,d}(\mathsf{q} \to \infty) \approx \mathcal{C}(m, d)\, \mathsf{q}^{-\alpha_{\mathrm{L}}/(\nu_{\mathrm{L}2}\theta)}. \tag{83}$$

From equation (82) we conclude that the universal coefficient $\mathcal{C}(m, d)$, to leading order in $1/n$, is given by

$$\mathcal{C}(m, d) = \mathcal{C}^{(0)}(m, d) + \mathrm{O}(1/n) \qquad \text{with } \mathcal{C}^{(0)}(m, d) = \frac{I(0, 1)}{I(1, 0)}. \tag{84}$$

In appendix B we show that the required two integrals can both be calculated explicitly for general values of m and d. The results can be found in equations (B.8) and (B.23). We refrain from giving the resulting rather lengthy expression for $\mathcal{C}(m, d)$ here, restricting ourselves to a discussion of some special cases of interest.

Consider, first, the uniaxial case $m = 1$. Here our result becomes

$$\mathcal{C}^{(0)}(1, d) = \frac{2^{-d+3/2}\sqrt{\pi}\,\Gamma\,[(2d-3)/4]\,\sin\,[(2\,d-5)\,\pi/4)]\,\left[2^d + 16\,\sin(d\,\pi/2)\right]}{\cos(d\,\pi)\Gamma(1/4)\,\Gamma[(d-1)/2]}. \tag{85}$$

As shown in figure 2, this coefficient is a smooth and finite function of d over the whole range $d_\ell = 5/2 \leqslant d \leqslant d^* = 9/2$. It has the ϵ expansion

$$\mathcal{C}^{(0)}(1, 9/2 - \epsilon) = 1 + (\pi - 4 - 2\ln 2)\,\frac{\epsilon}{12} + \mathrm{O}(\epsilon^2). \tag{86}$$

The corresponding results for the biaxial case $m = 2$ are

$$\mathcal{C}^{(0)}(2, d) = \pi^{-3/2}\cot(d\pi/2)\left[\Gamma[(5-d)/2]\,\Gamma[(d-4)/2] - 4^{5-d}\pi\,\frac{\Gamma(d-4)}{\Gamma(d-7/2)}\right] \tag{87}$$

and

$$\mathcal{C}^{(0)}(2, 5 - \epsilon) = 1 + (\ln 2 - 1)\,\epsilon + \mathrm{O}(\epsilon^2). \tag{88}$$

Again, this coefficient behaves smoothly between the upper and lower critical dimensions $d_\ell = 3$ and $d^* = 5$ (see figure 2).

9. Summary and discussion

In this paper we have shown how to utilize the $1/n$ expansion for the study of critical behaviour at m-axial LPs. We have been able to determine the $\mathrm{O}(1/n)$ corrections of the correlation exponents $\eta_{\mathrm{L}2}$ and $\eta_{\mathrm{L}4}$, and of the related anisotropy exponent θ, for general values of m and d with $0 \leqslant m \leqslant d$ and $2 + m/2 < d \leqslant 4 + m/2$. What makes such calculations a challenge is a combination of two problems: the anisotropic scale invariance one encounters already at the level of the free theory, and the complicated forms of the propagators' scaling functions at the LP.

To cope with these difficulties, it proved useful to employ a properly adjusted and generalized modification of the technique Vasiliev *et al* (1981a, 1981b) introduced to compute the critical exponents η and ν of the standard n-component $|\phi|^4$ model up to order n^{-2}. We believe that the present work may serve as a starting point for more ambitious studies based on the $1/n$ expansion. One question which might be systematically investigated in this manner is whether the predictions for scaling functions of anisotropic scale invariant systems made recently by Henkel (1997, 2002) have any significance in cases where fluctuation effects must not be ignored.

Apart from yielding insights into the feasibility of the approach to such an anisotropic scale invariant system, our work permits us to draw two interesting conclusions. First of all, it provides unequivocal evidence of the fact that the critical exponents $\eta_{\mathrm{L}2}$, $\eta_{\mathrm{L}4}$ and θ differ for $d < d^*$ from their classical values. Although this is in complete accord with what the available dimensionality expansions about the upper and lower critical dimensions tell us (see sections 5 and 6), it goes considerably beyond these results because no extrapolation in d is involved in the $1/n$ expansion. This means in particular that the anisotropy exponent θ of the three-dimensional uniaxial LPs should be nonclassical.

Another remarkable aspect of our results is the interesting variation of the $\mathrm{O}(1/n)$ coefficient of the exponent $\eta_{\mathrm{L}4}$ with d. As illustrated in figure 3 for the uniaxial case $m = 1$, it first decreases to small negative values as d drops below d^*, then becomes positive as d is lowered further, before it drops back to zero at the lower critical dimension d_ℓ. If we accept the plausible hypothesis that the correlation exponents are continuous functions of d, then it seems reasonable to expect a qualitatively similar behaviour of the exponent $\eta_{\mathrm{L}4}$. In other words, $\eta_{\mathrm{L}4}$ should change sign somewhere below d^*. In fact, there are other examples where the lowest nontrivial term of the dimensionality expansion about d^* is negative, whereas extrapolations of higher-order calculations yield a positive value in $d = 3$ dimensions. This happens for instance in the case of the Ising model with quenched random bond disorder. Here the dimensionality expansion of the correlation exponent η about $d = 4$ begins with a negative contribution $\sim(\sqrt{4-d})^2$ (see e.g. Jayaprakash and Katz 1977), whereas the extrapolation of higher-order calculations yields a positive value for this exponent at $d = 3$ (Pelissetto and Vicari 2000).

Let us close with a few cautionary remarks. Our results to $\mathrm{O}(1/n)$ are mathematically well defined over the whole regime where $2+m/2 < d \leqslant 4+m/2$ and $0 \leqslant m \leqslant d$ (the shaded region in figure 1). However, applying them to real three-dimensional systems requires some care. They should be combined with information from other sources such as dimensionality expansions, simulations, exact results and experiments, keeping in mind their limitations.

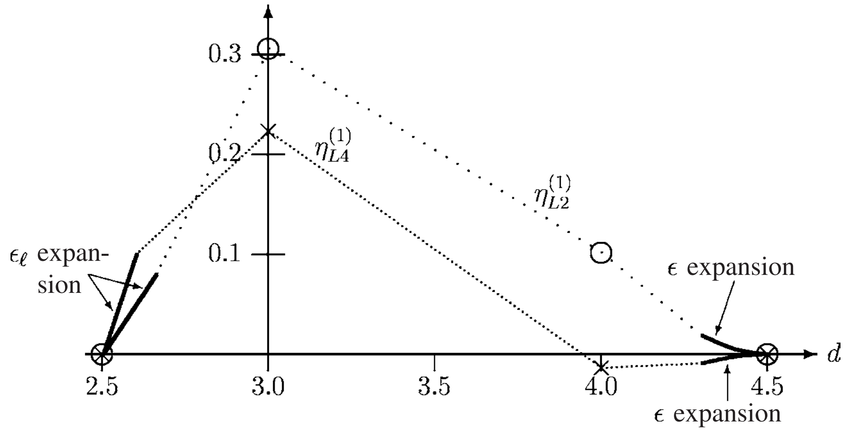


Figure 3. Behaviour of the $1/n$ coefficients $\eta_{L2}^{(1)}(1, d)$ (open circles) and $\eta_{L4}^{(1)}(1, d)$ (crosses) as functions of d. The four points displayed for each one of them correspond to results described in section 7. The thick lines near the upper and lower critical dimensions represent the limiting forms $\sim\epsilon^2$ and $\sim\epsilon_\ell$, which the dimensionality expansions mentioned in sections 5 and 6 yield. The widely or densely dotted lines serve to guide the eyes. The exponent $\eta_{L4}^{(1)}(1, d)$ has a small negative value beneath the upper critical dimension $d^*(1) = 9/2$ and appears to change sign somewhat below $d = 4$.

For example, depending on whether $m = 2$ or $m > 2$, the lower critical dimension d_ℓ is $d_\ell = 3$ or $d_\ell > 3$, respectively, whenever $n \geqslant 2$. Hence, unless the $O(n)$ symmetry is explicitly broken, no m-axial LPs are expected to occur at nonzero temperatures for $d = 3$, which leaves us with the case of uniaxial LPs in three dimensions. On the other hand, contributions to the Hamiltonian that break the $O(n)$ symmetry are generically expected, for example, for magnetic crystals. An analysis of such spin anisotropies can be found in Hornreich (1979).

For $m \geqslant 2$, one must also worry about anisotropies of another type: space anisotropies breaking the isotropy in the m-dimensional subspace. These give rise to corresponding anisotropic terms quadratic in $\partial^2\phi/\partial z_\alpha \partial z_{\alpha'}$, which we also have not taken into account here. Their effects have been investigated recently within the framework of the ϵ expansion (Diehl *et al* 2003b).

Last, but not least, let us mention that the technique of matching asymptotic scaling forms, on which our above analysis was based, can be extended to determine the $O(1/n)$ corrections of the thermal exponents ν_{L2}, ν_{L4}, γ_L and α_L for m-axial LPs (Shpot *et al* 2005).

Acknowledgments

We are indebted to S B Rutkevich for deriving the result (72) for the integral (70). One of us (MASh) thanks P A Hlushak and V B Solovyan for their help with Fortran programming in the numerical calculation involving this integral at a time when the simplified expression was not yet known.

We gratefully acknowledge support by the Deutsche Forschungsgemeinschaft (DFG)—in the initial phase via the Leibniz programme (grant No Di 378/2) and Sonderforschungsbereich 237, in the final phase via DFG grant No Di 378/3. One of us (YuP) also enjoyed support from the Russian Foundation for Basic Research (grant No 03-01-00837) and the Nordic Grant for Network Cooperation with the Baltic Countries and Northwest Russia FIN-20/2003. YuP and MASh thank Fachbereich Physik at the Universität Essen (now Universität Duisburg-Essen) for hospitality.

Appendix A. Large-n behaviour of momentum integrals

In subsequent calculations we shall need the properties of the following D-dimensional momentum integral:

$$J_D(f,t) = \int_{\boldsymbol{k}}^{(D)} k^{-2f}\, |\boldsymbol{k}+\mathbf{1}|^{-2t} = \mathcal{V}(f,t;D-f-t), \tag{A.1}$$

where the function $\mathcal{V}$ is defined by

$$\mathcal{V}(c_1,c_2;c_3) = (4\pi)^{-D/2} a(c_1)a(c_2)a(c_3), \qquad a(c_i) \equiv \frac{\Gamma\,(D/2-c_i)}{\Gamma(c_i)}, \qquad i=1,2,3. \tag{A.2}$$

Let us consider the parameters f and t of the integral (A.1) as regular functions of some small parameter κ. Choosing $l \geqslant 0$ to be integer, we set

$$f+t = \frac{D}{2} - l + \kappa. \tag{A.3}$$

Then the integral $J_D(f,t)$ is singular in the region of large k for small values of κ. Its singular part is given by

$$\mathcal{V}_{\text{sing}}(f,t;D/2+l-\kappa) = K_D \frac{(-1)^l}{l!} \frac{(f)_l\,(t)_l|_{\kappa=0}}{(D/2)_l} \frac{1}{2\kappa}, \tag{A.4}$$

where K_D was defined in equation (30), and $(\cdots)_l$ are Pochhammer symbols:

$$(w)_l \equiv \frac{\Gamma(w+l)}{\Gamma(w)} = w(w+1)\cdots(w+l-1), \qquad (w)_0 = 1. \tag{A.5}$$

Specifically for $l=0$, $l=1$, $l=2$ and $\kappa\to 0$, we have

$$\mathcal{V}_{\text{sing}}\,(f,t;D/2-\kappa) = \frac{K_D}{2\kappa}, \tag{A.6}$$

$$\mathcal{V}_{\text{sing}}\,(f,t;D/2+1-\kappa) = -\frac{K_D}{D\kappa} f\,t|_{\kappa=0}, \tag{A.7}$$

$$\mathcal{V}_{\text{sing}}\,(f,t;D/2+2-\kappa) = \frac{K_D}{D(D+2)\kappa} f(f+1)\,t(t+1)|_{\kappa=0}. \tag{A.8}$$

Two relevant examples of the integral $J_D(f,t)$ are

$$J_d(1,1) = \int_{\boldsymbol{p}}^{(d)} \frac{1}{p^2\,(\boldsymbol{p}+\mathbf{1})^2} = (4\pi)^{-d/2} \frac{\Gamma\,(2-d/2)}{\Gamma(d-2)} \Gamma^2(d/2-1) \tag{A.9}$$

and

$$J_d(2,2) = \int_{\boldsymbol{q}}^{(d)} \frac{1}{q^4\,(\boldsymbol{q}+\mathbf{1})^4} = (4\pi)^{-d/2} \frac{\Gamma\,(4-d/2)}{\Gamma(d-4)} \Gamma^2\left(\frac{d}{2}-2\right). \tag{A.10}$$

Now let us consider the momentum integrals $I_1(n)$ and $I_2(n)$ of equations (27) and (28). We have to isolate their pole contributions at $n=\infty$. For further convenience we write them as

$$I_1(n) = \int_{\boldsymbol{p}}^{(\bar{m})} \frac{p^{4\tilde{\Delta}_\phi-\bar{m}}}{|\boldsymbol{p}+\mathbf{1}|^{2\tilde{\Delta}_\phi}} \int_{\boldsymbol{q}}^{(m)} G_1\left(\frac{p^{2\theta}}{|\boldsymbol{p}+\mathbf{1}|^{2\theta}} q^2\right) \frac{1}{F(1,q)}, \tag{A.11}$$

$$I_2(n) = \int_{\boldsymbol{q}}^{(m)} \frac{q^{4\tilde{\Delta}_\phi/\theta-m}}{|\boldsymbol{q}+\mathbf{1}|^{2\tilde{\Delta}_\phi/\theta}} \int_{\boldsymbol{p}}^{(\bar{m})} G_2\left(\frac{q^{2/\theta}}{|\boldsymbol{q}+\mathbf{1}|^{2/\theta}} p^2\right) \frac{1}{F(p,1)}, \tag{A.12}$$

where $G_1(Q^2)$ and $G_2(P^2)$ denote the scaling functions $\tilde{G}_\phi^{(\text{as})}(1,Q)$ and $\tilde{G}_\phi^{(\text{as})}(P,1)$.

Adding and subtracting the asymptotic forms of the arguments of G_1 and G_2 for $p \to \infty$ and $q \to \infty$ we write these functions as

$$G_1\left(q^2\, p^{2\theta} |\mathbf{p}+\mathbf{1}|^{-2\theta}\right) \equiv G_1(q^2 + q^2\alpha_{\mathbf{p}}) = \sum_{s=0}^{2} \frac{1}{s!} \frac{d^s G_1(q^2)}{d(q^2)^s} q^{2s} \alpha_{\mathbf{p}}^s + R_1(p, q), \tag{A.13}$$

$$G_2\left(p^2\, q^{2/\theta} |\mathbf{q}+\mathbf{1}|^{-2/\theta}\right) \equiv G_2(p^2 + p^2\beta_{\mathbf{q}}) = \sum_{s=0}^{4} \frac{1}{s!} \frac{d^s G_2(p^2)}{d(p^2)^s} p^{2s} \beta_{\mathbf{q}}^s + R_2(p, q). \tag{A.14}$$

The deviations

$$\alpha_{\mathbf{p}} \equiv p^{2\theta} |\mathbf{p}+\mathbf{1}|^{-2\theta} - 1 \qquad \text{and} \qquad \beta_{\mathbf{q}} \equiv q^{2/\theta} |\mathbf{q}+\mathbf{1}|^{-2/\theta} - 1 \tag{A.15}$$

are of order p^{-1} and q^{-1} for large p and q. Including their successive powers enhances the ultraviolet convergence of p and q integrations in equations (A.11) and (A.12). The remainders $R_1(p, q)$ and $R_2(p, q)$ of the Taylor expansions contain sufficiently high powers of $\alpha_{\mathbf{p}}$ and $\beta_{\mathbf{q}}$ to make the integrations finite as $n \to \infty$. These convergent contributions would be required for a calculation to $\mathrm{O}(n^{-2})$. All we need now is

$$I_1(n) = \sum_{s=0}^{2} \frac{J_s^\alpha}{s!} \lim_{n\to\infty} \int_{\mathbf{q}}^{(m)} q^{2s} \frac{d^s G_1(q^2)}{d(q^2)^s} \frac{1}{F(1, q)} + \mathrm{O}(n^0), \tag{A.16}$$

$$I_2(n) = \sum_{s=0}^{4} \frac{J_s^\beta}{s!} \lim_{n\to\infty} \int_{\mathbf{p}}^{(\bar{m})} p^{2s} \frac{d^s G_2(p^2)}{d(p^2)^s} \frac{1}{F(p, 1)} + \mathrm{O}(n^0). \tag{A.17}$$

The coefficients J_s^α and J_s^β are given by the integrals

$$J_s^\alpha = \int_{\mathbf{p}}^{(\bar{m})} p^{4\tilde{\Delta}_\phi - \bar{m}}\, |\mathbf{p}+\mathbf{1}|^{-2\tilde{\Delta}_\phi}\, \alpha_{\mathbf{p}}^s, \qquad s = 0, 1, 2, \tag{A.18}$$

$$J_s^\beta = \int_{\mathbf{q}}^{(m)} q^{\frac{4\tilde{\Delta}_\phi}{\theta} - m}\, |\mathbf{q}+\mathbf{1}|^{-\frac{2\tilde{\Delta}_\phi}{\theta}}\, \beta_{\mathbf{q}}^s, \qquad s = 0, \ldots, 4, \tag{A.19}$$

which have simple poles in $1/n$ at $n = \infty$. This fact allows us to take the limit $n \to \infty$ in the remaining integrals over $\mathbf{q}$ and $\mathbf{p}$ in equations (A.16) and (A.17).

Making binomial expansions of the integer powers $\alpha_{\mathbf{p}}^s$ and $\beta_{\mathbf{q}}^s$ in equations (A.18) and (A.19), we can express J_s^α and J_s^β in terms of the integral (A.1):

$$J_s^\alpha = \sum_{j=0}^{s} (-1)^{s-j} C_s^j \mathcal{V}\left(\frac{\bar{m}}{2} - 2\tilde{\Delta}_\phi - j\theta, \tilde{\Delta}_\phi + j\theta; \frac{\bar{m}}{2} + 1 - \frac{\eta_{\mathrm{L2}}^{(1)}}{2n}\right), \tag{A.20}$$

$$J_s^\beta = \sum_{j=0}^{s} (-1)^{s-j} C_s^j \mathcal{V}\left[\frac{1}{\theta}\left(\frac{m\theta}{2} - 2\tilde{\Delta}_\phi - j\right), \frac{\tilde{\Delta}_\phi + j}{\theta}; \frac{m}{2} + 2 - \frac{\eta_{\mathrm{L2}}^{(1)} + 4\theta^{(1)}}{n}\right], \tag{A.21}$$

where C_s^j are the corresponding binomial coefficients.

Comparing the arguments of the function $\mathcal{V}$ in equations (A.20) and (A.21) with their counterparts in equation (A.4), we read off $l = 1$ and $\kappa = \eta_{\mathrm{L2}}^{(1)}/(2n)$ for J_s^α, and $l = 2$ with $\kappa = (\eta_{\mathrm{L2}}^{(1)} + 4\theta^{(1)})/n$ for J_s^β. With the aid of the results (A.7) and (A.8) we then obtain the required leading large-n behaviour of J_s^α and J_s^β:

$$J_s^\alpha = \frac{K_{\bar{m}}}{\bar{m}} \frac{2n}{\eta_{\mathrm{L2}}^{(1)}} A_s + \mathrm{O}(n^0), \tag{A.22}$$

$$J_s^\beta = \frac{K_m}{m(m+2)} \frac{n}{\eta_{\mathrm{L2}}^{(1)} + 4\theta^{(1)}} B_s + \mathrm{O}(n^0). \tag{A.23}$$

Here A_s (with $s = 0, 1, 2$) and B_s (with $s = 0, \ldots, 4$) denote the sums

$$A_s = \sum_{j=0}^{s} (-1)^{s-j+1} C_s^j \left(\frac{\bar{m}}{2} - 2 - \frac{j}{2}\right)\left(1 + \frac{j}{2}\right), \tag{A.24}$$

$$B_s = \sum_{j=0}^{s} (-1)^{s-j} C_s^j \left(\frac{m}{2} - 4 - 2j\right)\left(\frac{m}{2} - 3 - 2j\right)(2+2j)(3+2j). \tag{A.25}$$

A simple calculation gives

$$\begin{aligned} &A_0 = \tfrac{1}{2}(4-d+m), && A_1 = \tfrac{1}{4}(7-d+m), && A_2 = \tfrac{1}{2};\\ &B_0 = \tfrac{3}{2}(8-m)(6-m), && B_1 = \tfrac{1}{2}(16-m)(66-7m), && \\ &B_2 = 2(612 - 58m + m^2), && B_3 = 48(24-m), && B_4 = 384. \end{aligned} \tag{A.26}$$

In conjunction with the relations (A.22) and (A.23), these results give us the coefficients J_s^α and J_s^β in equations (A.16) and (A.17).

The limit $n \to \infty$ in these equations reduces the scaling functions in the integrands to those of the free theory. That is, we may use the large-n result

$$\lim_{n\to\infty} F(p, q) = I(p, q) = \int_{k'}^{(d)} \frac{1}{(p'^2 + q'^4)\left(|\boldsymbol{p}' + \boldsymbol{p}|^2 + |\boldsymbol{q}' + \boldsymbol{q}|^4\right)} \tag{A.27}$$

for both scaling functions $F(1, q)$ and $F(p, 1)$ introduced in equation (24).

Similarly, the scaling functions $G_1(q^2)$ and $G_2(p^2)$ in equations (A.16) and (A.17) reduce to

$$\tilde{G}_1^{(0)}(q^2) = \frac{1}{1+q^4} \qquad \text{and} \qquad \tilde{G}_2^{(0)}(p^2) = \frac{1}{p^2+1} \tag{A.28}$$

in this limit.

We can now evaluate the sums in equations (A.16) and (A.17). Apart from overall factors, we have

$$\sum_{s=0}^{2} \frac{A_s}{s!} q^{2s} \frac{d^s G_1^{(0)}(q^2)}{d(q^2)^s} = \frac{\mathcal{P}_1(q^4)}{2(1+q^4)^3}, \tag{A.29}$$

$$\sum_{s=0}^{4} \frac{B_s}{s!} p^{2s} \frac{d^s G_2^{(0)}(p^2)}{d(p^2)^s} = \frac{\mathcal{P}_2(p^2)}{2(1+p^2)^5}, \tag{A.30}$$

where the polynomials $\mathcal{P}_1(q^4)$ and $\mathcal{P}_2(p^2)$ are given explicitly in equations (31) and (35) of the main text. Finally, equations (A.16) and (A.17), along with (A.22), (A.23) and (A.29), (A.30), yield the leading-order expressions of $I_1(n)$ and $I_2(n)$ given in equations (29) and (34).

Appendix B. The one-loop integral

The one-loop Feynman integral $I(p, q)$ is an important ingredient of our calculations. Unfortunately, we did not succeed in evaluating it for generic dimensions and external momenta. Various simplified expressions which result for special values of m and d are presented in the main text. Below, we consider the simplifying cases of zero external momenta where closed analytic expressions are obtained for arbitrary m and d.

B.1. The integral $I(p,0)$

Consider the integral

$$I(p,0)=\int_{p'}^{(\bar m)}\int_{q'}^{(m)}\frac{1}{p'^2+q'^4}\,\frac{1}{(\boldsymbol{p}'+\boldsymbol{p})^2+q'^4}. \tag{B.1}$$

It is convenient to convert this integral to the coordinate representation where it is given by a Fourier transform of the squared free propagator:

$$I(p,0)=\int \mathrm{d}^{\bar m}r\int \mathrm{d}^m z\big[G_\phi^{(0)}(r,z)\big]^2\mathrm{e}^{-\mathrm{i}\boldsymbol{pr}}. \tag{B.2}$$

Using the scaling representation (63) for $G_\phi^{(0)}(r,z)$ and changing the integration variable z via $z=\upsilon\sqrt{r}$ we obtain

$$I(p,0)=\int \mathrm{d}^{\bar m}r\; r^{-4+m/2+2\epsilon}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{pr}}\int \mathrm{d}^m\upsilon\;\Phi^2(\upsilon). \tag{B.3}$$

The r integration is standard, giving

$$\int \mathrm{d}^{\bar m}r r^{-4+m/2+2\epsilon}\mathrm{e}^{-\mathrm{i}\boldsymbol{pr}}=2^{\epsilon}\,\pi^{\bar m/2}\frac{\Gamma(\epsilon/2)}{\Gamma(2-m/4-\epsilon)}p^{-\epsilon}. \tag{B.4}$$

To calculate the integral over υ, we use the Fourier representation (Diehl and Shpot 2000, Shpot and Diehl 2001)

$$\Phi(\upsilon)=(2\pi)^{-\bar m/2}\int_q^{(m)} q^{\bar m-2}\,K_{\bar m/2-1}(q^2)\,\mathrm{e}^{\mathrm{i}\boldsymbol{q}\boldsymbol{\upsilon}}. \tag{B.5}$$

This leads us to

$$\int \mathrm{d}^m\upsilon\;\Phi^2(\upsilon)=(2\pi)^{-\bar m}\int_q^{(m)} q^{2\bar m-4}K^2_{\bar m/2-1}(q^2). \tag{B.6}$$

The last integral is known for arbitrary m and d. We obtain

$$\int \mathrm{d}^m\upsilon\Phi^2(\upsilon)=(2\pi)^{-d}\,\pi^{m/2}\frac{1}{2}2^{-\epsilon}\,\Gamma^2(1-\epsilon/2)\,\frac{\Gamma(2-m/4-\epsilon)}{\Gamma(2-\epsilon)}\,\frac{\Gamma(m/4)}{\Gamma(m/2)}. \tag{B.7}$$

Multiplying the two contributions yields

$$I(p,0)=(4\pi)^{-d/2}\,\frac{1}{2}\,\Gamma(\epsilon/2)\,\frac{\Gamma^2(1-\epsilon/2)}{\Gamma(2-\epsilon)}\,\frac{\Gamma(m/4)}{\Gamma(m/2)}p^{-\epsilon}. \tag{B.8}$$

B.2. The integral $I(0,q)$

Let us consider another special case of the one-loop integral $I(p,q)$,

$$I(0,1)=\int_p^{(\bar m)}\int_q^{(m)}\frac{1}{p^2+q^4}\,\frac{1}{p^2+(\boldsymbol{q}+\boldsymbol{1})^4}. \tag{B.9}$$

Its treatment is somewhat more involved. We shall use some tricks employed by Mergulhão and Carneiro (1999) in a similar calculation. Making a partial fraction expansion

$$\frac{1}{(a+b)(a+c)}=\frac{1}{c-b}\left(\frac{1}{a+b}-\frac{1}{a+c}\right), \tag{B.10}$$

we represent (B.9) as

$$I(0,1)=\int_q^{(m)}\frac{1}{(\boldsymbol{q}+\boldsymbol{1})^4-q^4}\int_p^{(\bar m)}\left[\frac{1}{p^2+q^4}-\frac{1}{p^2+(\boldsymbol{q}+\boldsymbol{1})^4}\right]. \tag{B.11}$$

Integrating over p gives

$$I(0,1) = (4\pi)^{-\bar{m}/2}\Gamma(-\nu)\int_q^{(m)} \frac{1}{(\boldsymbol{q}+\boldsymbol{1})^4 - q^4}\left(q^{4\nu} - |\boldsymbol{q}+\boldsymbol{1}|^{4\nu}\right) \tag{B.12}$$

with

$$\nu = \frac{\bar{m}}{2} - 1 = 1 - \frac{m}{4} - \frac{\epsilon}{2}. \tag{B.13}$$

It is useful to represent the difference in the last brackets via the elementary integral

$$a^\alpha - b^\alpha = -\alpha\,(b-a)\int_0^1 \frac{\mathrm{d}x}{[a + x(b-a)]^{1-\alpha}} \tag{B.14}$$

with $a = q^2$, $b = (\boldsymbol{q}+\boldsymbol{1})^2$ and $\alpha = 2\nu$. The factor $(b-a)$ cancels with a corresponding term of the denominator (see equation (B.12)). We obtain

$$I(0,1) = \frac{(-2\nu)\,\Gamma(-\nu)}{(4\pi)^{\bar{m}/2}}\int_q^{(m)} \frac{1}{q^2 + (\boldsymbol{q}+\boldsymbol{1})^2}\int_0^1 \mathrm{d}x\,\frac{1}{\left(q^2 + 2\boldsymbol{q}\boldsymbol{1}x + x\right)^{1-2\nu}}. \tag{B.15}$$

Here, we represent the momentum-dependent factors through Laplace integrals

$$a^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty \mathrm{d}x\,x^{\alpha-1}\mathrm{e}^{-ax}. \tag{B.16}$$

This yields

$$I(0,1) = \frac{(-2\nu)\,\Gamma(-\nu)}{(4\pi)^{\bar{m}/2}\,\Gamma(1-2\nu)} \times \int_0^1 \mathrm{d}x\int_0^\infty \mathrm{d}y\int_0^\infty \mathrm{d}z\,z^{-2\nu}\int_q^{(m)} \mathrm{e}^{-(2q^2+2\boldsymbol{q}\boldsymbol{1}+1)y}\,\mathrm{e}^{-(q^2+2\boldsymbol{q}\boldsymbol{1}x+x)z}. \tag{B.17}$$

Performing the Gaussian integration over q we get

$$I(0,1) = \frac{(-2\nu)\,\Gamma(-\nu)}{(4\pi)^{d/2}\,\Gamma(1-2\nu)}\int_0^1 \mathrm{d}x\int_0^\infty \mathrm{d}y\int_0^\infty \mathrm{d}z\,\frac{z^{-2\nu}}{(2y+z)^{m/2}}\,\mathrm{e}^{-\frac{y^2 - x^2z^2 + xz^2 + yz}{2y+z}}. \tag{B.18}$$

The integrals over t and z can be decoupled by a rescaling $y = zt$ of the integration variable y. This gives

$$I(0,1) = \frac{(-2\nu)\,\Gamma(-\nu)}{(4\pi)^{-d/2}\,\Gamma(1-2\nu)}\int_0^1 \mathrm{d}x\int_0^\infty \frac{\mathrm{d}t}{(2t+1)^{m/2}}\int_0^\infty \mathrm{d}z\,z^{-1+\epsilon}\,\mathrm{e}^{-z\frac{t^2+t+x-x^2}{2t+1}}. \tag{B.19}$$

The z integral is of the form (B.16). We obtain

$$I(0,1) = \frac{(-2\nu)\,\Gamma(-\nu)\Gamma(\epsilon)}{(4\pi)^{d/2}\,\Gamma(1-2\nu)}\int_0^1 \mathrm{d}x\int_0^\infty \frac{\mathrm{d}t}{(2t+1)^{m/2-\epsilon}}\,(t^2 + t + x - x^2)^{-\epsilon}. \tag{B.20}$$

Inside of the last brackets we add and subtract $1/4$. Next, we introduce the new integration variables $y = 2t+1$ and $s = 2x-1$. A little bit of algebra then yields

$$I(0,1) = \frac{(-2\nu)\,\Gamma(-\nu)\,\Gamma(\epsilon)}{(4\pi)^{d/2}\,\Gamma(1-2\nu)}\,\frac{2^{2\epsilon}}{4}\int_{-1}^1 \mathrm{d}s\int_1^\infty \mathrm{d}y\,y^{-m/2+\epsilon}(y^2 - s^2)^{-\epsilon}. \tag{B.21}$$

Here the last integration produces a Gauss hypergeometric function:

$$I(0,1) = \frac{(-2\nu)\,\Gamma(-\nu)\,\Gamma(\epsilon)}{(4\pi)^{d/2}\,\Gamma(1-2\nu)}\,\frac{2^{2\epsilon-1}}{m-2+2\epsilon}\int_{-1}^1 \mathrm{d}s\,{}_2F_1\left(\epsilon, \tfrac{m-2+2\epsilon}{4}; \tfrac{2+m+2\epsilon}{4}; s^2\right). \tag{B.22}$$

Performing the remaining integration we finally obtain

$$I(0,q) = q^{-2\epsilon}\,(4\pi)^{-d/2}\,\frac{\Gamma(-\nu)}{\Gamma(1-2\nu)}\,\Gamma(\epsilon)\Gamma(1-\epsilon)\,2^{-2+2\epsilon} \times \left[\frac{\sqrt{\pi}}{\Gamma(3/2-\epsilon)} + \frac{4}{2-m-2\epsilon}\,\frac{\Gamma[(2+m+2\epsilon)/4]}{\Gamma[(2+m-2\epsilon)/4]}\right], \tag{B.23}$$

where we reintroduced the initially suppressed trivial dependence on q, taking into account that $\nu = 1 - m/4 - \epsilon/2$.

References

Abe R 1973 Expansion of a critical exponent in inverse powers of spin dimensionality *Prog. Theor. Phys.* **49** 113

Abe R and Hikami S 1973 Critical exponents and scaling relations in $1/n$ expansion *Prog. Theor. Phys.* **49** 442

Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (*Applied Mathematics Series*) (Washington, DC: National Bureau of Standards)

Aharony O, Gubser S S, Maldacena J, Ooguri H and Oz Y 2000 Large N field theories, string theory and gravity *Phys. Rep.* **323** 183

Arnold P and Tomášik B 2000 T_c for dilute Bose gases: Beyond leading order in $1/N$ *Phys. Rev.* A **62** 063604

Bardeen W A, Lee B W and Shrock R E 1976 Phase transition in the nonlinear σ model in a $2+\epsilon$ continuum *Phys. Rev.* D **14** 985

Baym G, Blaizot J-P and Zinn-Justin J 2000 The transition temperature of the dilute interacting Bose gas for n internal states *Europhys. Lett.* **49** 150

Bervillier C 2004 Exact renormalization group equation for the Lifshitz critical point *Phys. Lett.* A **331** 110

Bray A 2002 Theory of phase ordering kinetics *Adv. Phys.* **51** 481

Brazovskiĭ S A 1975 Phase transition of an isotropic system to a nonuniform state *Sov. Phys.—JETP* **41** 85

Brézin E (ed) 1993 *The Large N-expansion in Quantum Field Theory and Statistical Physics: from Spin Systems to 2-dimensional Gravity* (Singapore: World Scientific)

Brézin E, Le Guillou J C and Zinn-Justin J 1976 Field theoretical approach to critical phenomena *Phase Transitions and Critical Phenomena* vol 6, ed C Domb and M S Green (London: Academic) chapter 3, pp 125–247

Brézin E and Zinn-Justin J 1976a Renormalization of the nonlinear σ model in $2+\epsilon$ dimensions—application to the Heisenberg ferromagnets *Phys. Rev. Lett.* **36** 691

Brézin E and Zinn-Justin J 1976b Spontaneous breakdown of continuous symmetries near two dimensions *Phys. Rev.* B **14** 3110

Campostrini M, Pelissetto A, Rossi P and Vicari E 1998 Two-point correlation function of three-dimensional $O(N)$ models: the critical limit and anisotropy *Phys. Rev.* E **57** 184

de Albuquerque L C and Leite M M 2001 Anisotropic Lifshitz point at $O(\epsilon_L^2)$ *J. Phys. A: Math. Gen.* **34** L327

Diehl H W 2002 Critical behavior at m-axial Lifshitz points *Proc. 5th Int. Conf. Renormalization Group 2002, (Tatranska Strba, High Tatra Mountains Slovakia, March 2002)*; *Acta Phys. Slovaca* **52** 271 (*Preprint* cond-mat/0205284)

Diehl H W 2004 Bulk and boundary critical behavior at Lifshitz points *Proc. 22nd Int. Conf. on Statistical Physics (STATPHYS 22) of the International Union of Pure and Applied Physics (IUPAP), (Bangalore, July 2004)*; *Pramana J. Phys.* **64** 1 (*Preprint* cond-mat/0407352) at press

Diehl H W, Gerwinski A and Rutkevich S 2003a Boundary critical behavior at m-axial Lifshitz points for a boundary plane parallel to the modulation axes *Phys. Rev.* B **68** 224428 (*Preprint* cond-mat/0308483)

Diehl H W and Shpot M 2000 Critical behavior at m-axial Lifshitz points: Field-theory analysis and ϵ-expansion results *Phys. Rev.* B **62** 12338 (*Preprint* cond-mat/0006007)

Diehl H W and Shpot M 2001 Lifshitz-point critical behaviour to $O(\epsilon^2)$ *J. Phys. A: Math. Gen.* **34** 9101 (*Preprint* cond-mat/0106502)

Diehl H W and Shpot M 2003 Comment on renormalization-group picture of the Lifshitz critical behavior *Phys. Rev.* B **68** 066401 (*Preprint* cond-mat/0305131)

Diehl H W, Shpot M A and Zia R K P 2003b Relevance of space anisotropy in the critical behavior of m-axial Lifshitz points *Phys. Rev.* B **68** 224415 (*Preprint* cond-mat/0307355)

Doherty J P, Moore M A, Kim J M and Bray A J 1994 Generalizations of the Kardar–Parisi–Zhang equation *Phys. Rev. Lett.* **72** 2041

Domb C and Green M S (ed) 1976 *Phase Transitions and Critical Phenomena* vol 6 (London: Academic)

Düchs D, Ganesan G, Fredrickson G H and Schmid F 2003 Fluctuation effects in ternary AB + A + B polymeric emulsions *Macromolecules* **36** 9237

Fisher M E 1983 Scaling, universality and renormalization group theory *Critical Phenomena* (*Springer Lecture Notes in Physics*) vol 186, ed F J W Hahne (Berlin: Springer) pp 1–139

Frachebourg L and Henkel M 1993 Exact correlation function at the Lifshitz points of the spherical model *Physica* A **195** 577

Franz M, Tešanović Z and Vafek O 2002 QED_3 theory of pairing pseudogap in cuprates: From d-wave superconductor to antiferromagnet via an algebraic Fermi liquid *Phys. Rev.* B **66** 054535

Garel T and Pfeuty P 1976 Commensurability effects on the critical behaviour of systems with helical ordering *J. Phys. C: Solid State Phys.* **9** L246

Gracey J A 2002a Critical exponent ω at $O(1/N)$ in $O(N) \times O(M)$ spin models *Nucl. Phys.* B **644** 433

Gracey J A 2002b Critical exponents in $O(N) \times O(M)$ spin models at $O(1/N^2)$ *Phys. Rev.* B **66** 134402

Grest G S and Sak J 1978 Low-temperature renormalization group for the Lifshitz point *Phys. Rev.* B **17** 3607

Guida R and Zinn-Justin J 1998 Critical exponents of the N-vector model *J. Phys. A: Math. Gen.* **31** 8103

Halperin B I, Hohenberg P C and Ma S-K 1972 Calculation of dynamic critical properties using Wilson's expansion methods *Phys. Rev. Lett.* **29** 1548

Henkel M 1997 Local scale invariance and strongly anisotropic equilibrium critical systems *Phys. Rev. Lett.* **78** 1940

Henkel M 2002 Phenomenology of local scale invariance: from conformal invariance to dynamical scaling *Nucl. Phys.* B **641** 405

Hornreich R M 1979 Renormalization-group analysis of critical modes of a ferromagnet *Phys. Rev.* B **19** 5914

Hornreich R M 1980 The Lifshitz point: phase diagrams and critical behavior *J. Magn. Magn. Mater.* **15–18** 387

Hornreich R M and Bruce A D 1978 Behaviour of the critical wavevector near a Lifshitz point *J. Phys. A: Math. Gen.* **11** 595

Hornreich R M, Luban M and Shtrikman S 1975a Critical behavior at the onset of $\vec{k}$-space instability on the λ line *Phys. Rev. Lett.* **35** 1678

Hornreich R M, Luban M and Shtrikman S 1975b Critical exponents at a Lifshitz point to $O(1/n)$ *Phys. Lett.* A **55** 269

Hornreich R M, Luban M and Shtrikman S 1977 Exactly solvable model exhibiting a multicritical point *Physica* A **86** 465

Inayat-Hussain A A and Buckingham M J 1990 Continuously varying critical exponents to $O(1/n)$ *Phys. Rev.* A **41** 5394

Jayaprakash C and Katz H J 1977 Higher-order corrections to the $\epsilon^{1/2}$ expansion of the critical behavior of the random Ising system *Phys. Rev.* B **16** 3987

Kalok L and Obermair G M 1976 Competing interaction and critical properties in the spherical model *J. Phys. C: Solid State Phys.* **9** 819

Le Doussal P and Wiese K J 2003 Functional renormalization group at large N for disordered elastic systems, and relation to replica symmetry breaking *Phys. Rev.* B **68** 174202

Le Doussal P and Wiese K J 2004 Derivation of the functional renormalization group β-function at order $1/N$ for manifolds pinned by disorder *Nucl. Phys.* B **701** 409

Leite M M 2003 Renormalization-group picture of the Lifshitz critical behavior *Phys. Rev.* B **67** 104415

Ma S-K 1973 Critical exponents above T_c to $O(1/n)$ *Phys. Rev.* A **7** 2172

McAvity D M and Osborn H 1995 Conformal field theories near a boundary in general dimensions *Nucl. Phys.* B **455** 522

Mergulhão C Jr and Carneiro C E I 1999 Field-theoretic calculation of critical exponents for the Lifshitz point *Phys. Rev.* B **59** 13954

Moshe M and Zinn-Justin J 2003 Quantum field theory in the large N limit: a review *Phys. Rep.* **385** 69

Mukamel D 1977 Critical behaviour associated with helical order near a Lifshitz point *J. Phys. A: Math. Gen.* **10** L249

Mukamel D and Luban M 1978 Critical behavior at a Lifshitz point: Calculation of a universal amplitude ratio *Phys. Rev.* B **18** 3631

Nicoll J F, Tuthill G F, Chang T S and Stanley H E 1976 Renormalization group calculation for critical points of higher order with general propagator *Phys. Lett.* A **58** 1

Parisi G 1980 Field-theoretic approach to second-order phase transitions in two- and three-dimensional systems *J. Stat. Phys.* **23** 49

Pelissetto A, Rossi P and Vicari E 2001 Large-n critical behavior of $O(n) \times O(m)$ spin models *Nucl. Phys.* B **607** 605

Pelissetto A and Vicari E 2000 Randomly dilute spin models: A six-loop field-theoretic study *Phys. Rev.* B **62** 6393

Pelissetto A and Vicari E 2002 Critical phenomena and renormalization group theory *Phys. Rep.* **368** 549

Polyakov A M 1975 Interaction of Goldstone particles in two dimensions: Applications to ferromagnets and Yang–Mills fields *Phys. Lett.* B **59** 79

Privman V, Hohenberg P C and Aharony A 1991 Universal critical-point amplitude relations *Phase Transitions and Critical Phenomena* vol 14, ed C Domb and J L Lebowitz (London: Academic) pp 1–134

Sak J and Grest G S 1978 Critical exponents for the Lifshitz point: epsilon expansion *Phys. Rev.* B **17** 3602

Selke W 1992 Spatially modulated structures in systems with competing interactions *Phase Transitions and Critical Phenomena* vol 15, ed C Domb and J L Lebowitz (London: Academic) pp 1–72

Shpot M and Diehl H W 2001 Two-loop renormalization-group analysis of critical behavior at m-axial Lifshitz points *Nucl. Phys.* B **612** 340 (*Preprint* cond-mat/0106105)

Shpot M A, Pis'mak Yu M and Diehl H W 2005 unpublished

Vasiliev A N 1998 *Quantum Field Renormalization Group in the Theory of Critical Behavior and Stochastic Dynamics* 1st edn (St-Petersburg: PINF Publ.) (in Russian); 2004 *The Field Theoretic Renormalization Group in Critical Behavior Theory and Stochastic Dynamics* (London: Chapman and Hall) (Engl. Transl.)

Vasiliev A N, Pis'mak Y M and Khonkonen Y R 1981a Simple method of calculating critical indices in the $1/n$ expansion *Teor. Mat. Fiz.* **46** 157

Vasiliev A N, Pis'mak Y M and Khonkonen Y R 1981b $1/n$-expansion: calculation of η and ν in order $1/n^2$ for arbitrary dimension *Teor. Mat. Fiz.* **47** 291

Vasiliev A N, Pis'mak Y M and Khonkonen Y R 1981c $1/n$-expansion: calculation of η in the order $1/n^3$ by conformal bootstrap method *Teor. Mat. Fiz.* **50** 127

Wilson K G and Fisher M E 1972 Critical exponents in 3.99 dimensions *Phys. Rev. Lett.* **28** 240

Wilson K G and Kogut J 1974 The renormalization group and the ϵ expansion *Phys. Rep.* C **12** 75

Zinn-Justin J 1996 *Quantum Field Theory and Critical Phenomena* (*International Series of Monographs on Physics*) 3rd edn (Oxford: Clarendon) pp 618–21